

CS103  
FALL 2025



# Lecture 12: Mathematical Induction

Part 1 of 2

# Quick Announcements!

# Problem Set Four

- Problem Set Four is due this Friday at 1:00PM.
  - It's smaller than usual.
- PS3 solutions are posted!
- Exam grading is this Saturday.

Okay, let's kick off our exploration of today's material with some kinetic activity.

Let's do the wave!

# The Wave

- If done properly, everyone will eventually end up joining in.
- Why is that? There are two primary components:
  - Someone (me!) started everyone off.
  - Once the person before you did the wave, you did the wave.

Let  $P$  be some predicate. The **principle of mathematical induction** states that if

If it starts  
true...

$P(0)$  is true

and

$\forall k \in \mathbb{N}. (P(k) \rightarrow P(k+1))$

then

$\forall n \in \mathbb{N}. P(n)$

...then it's  
always true.

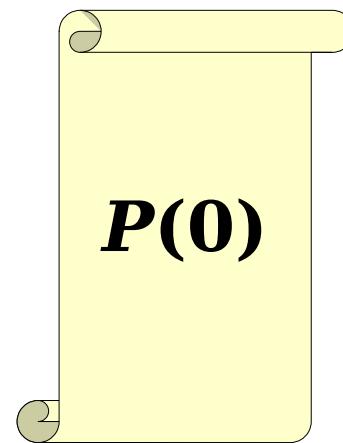
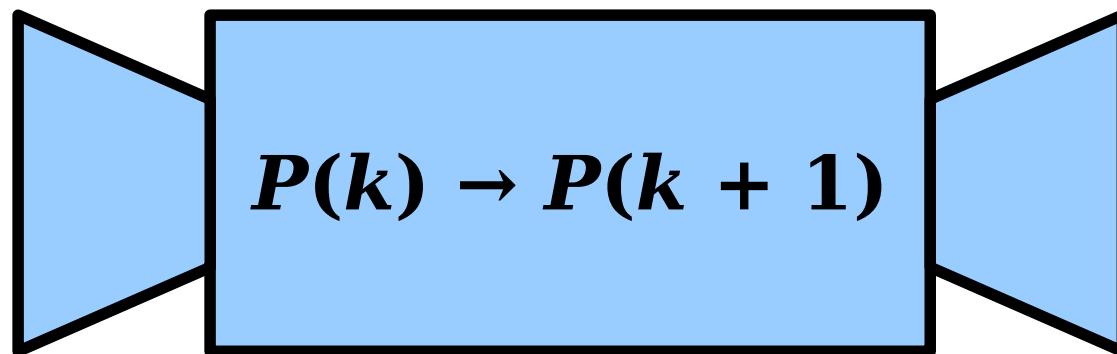
# Induction, Intuitively

$P(0)$

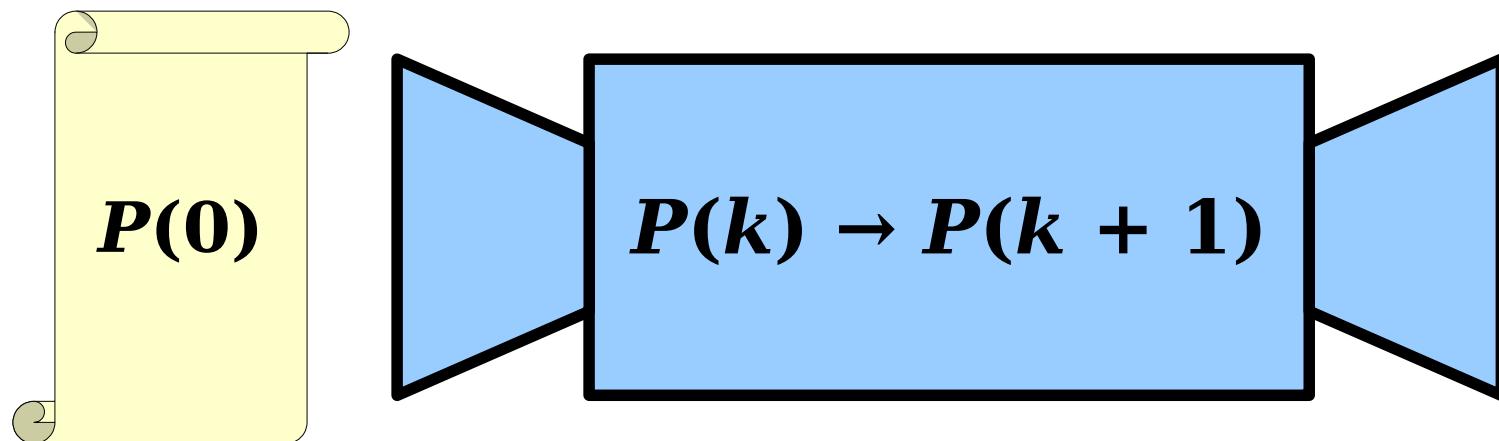
$\forall k \in \mathbb{N}. (P(k) \rightarrow P(k+1))$

- It's true for 0.
- Since it's true for 0, it's true for 1.
- Since it's true for 1, it's true for 2.
- Since it's true for 2, it's true for 3.
- Since it's true for 3, it's true for 4.
- Since it's true for 4, it's true for 5.
- Since it's true for 5, it's true for 6.
- ...

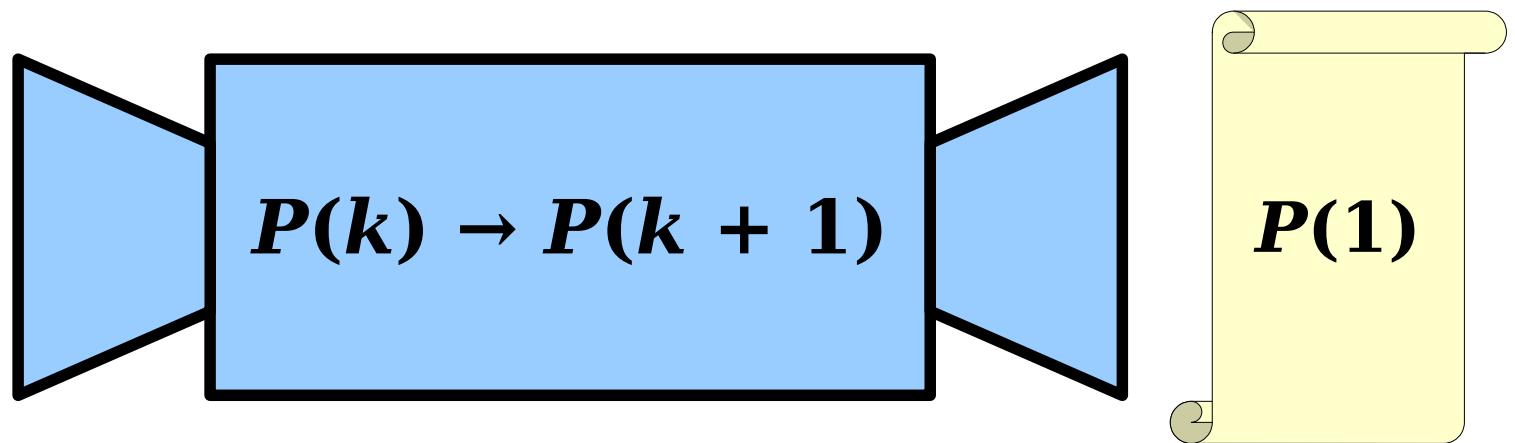
# Why Induction Works



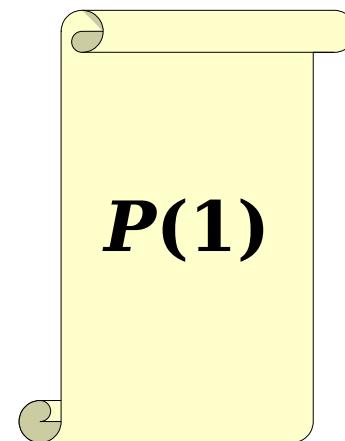
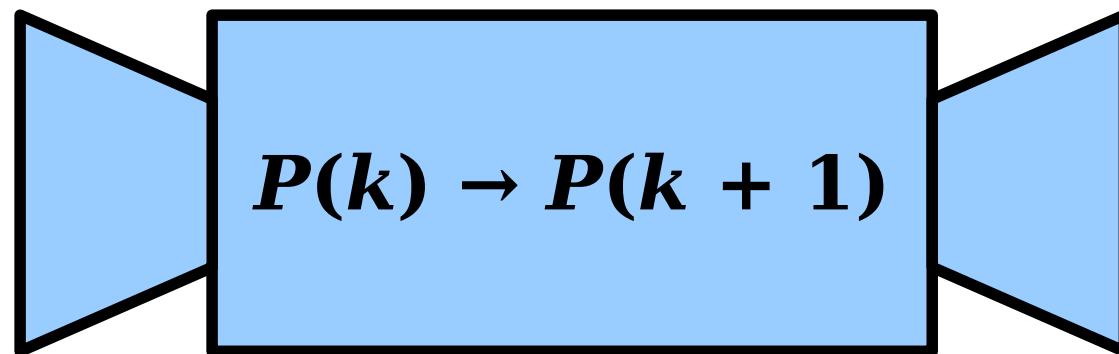
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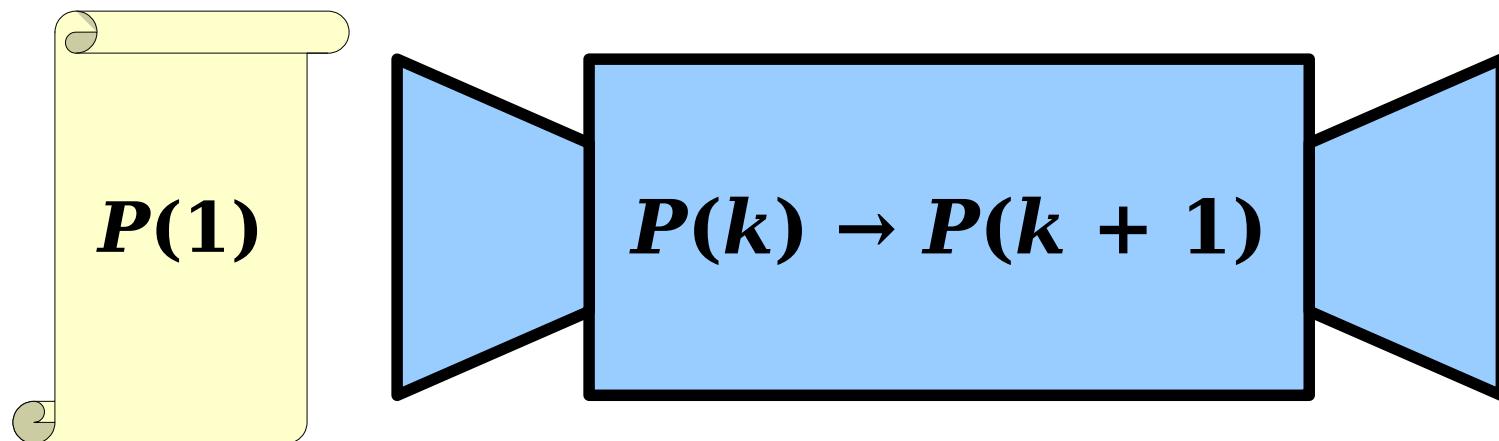
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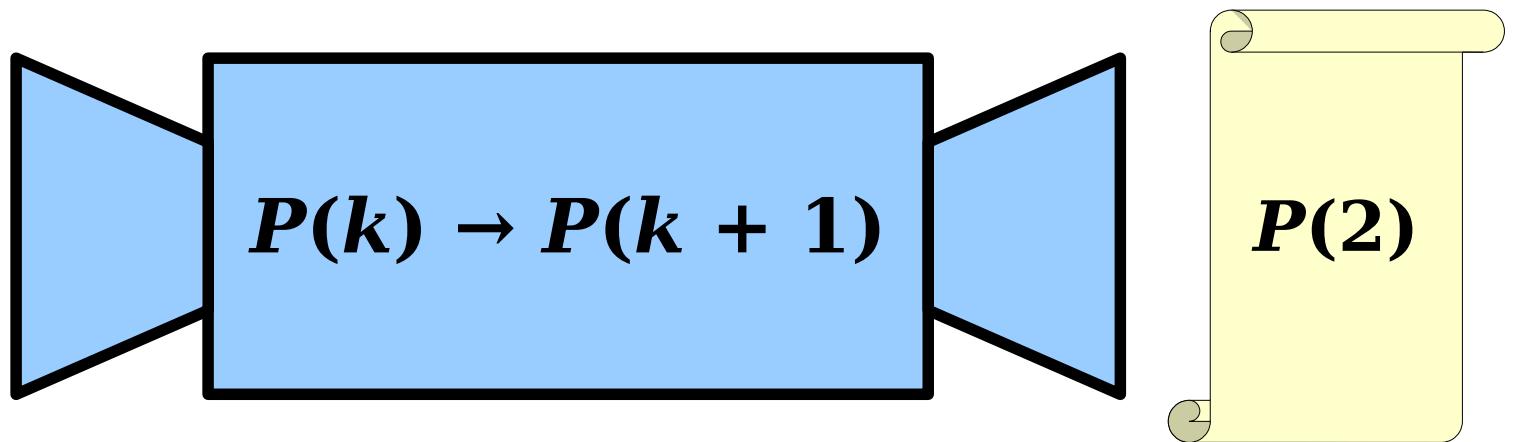
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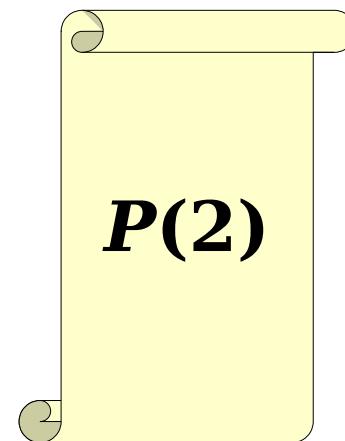
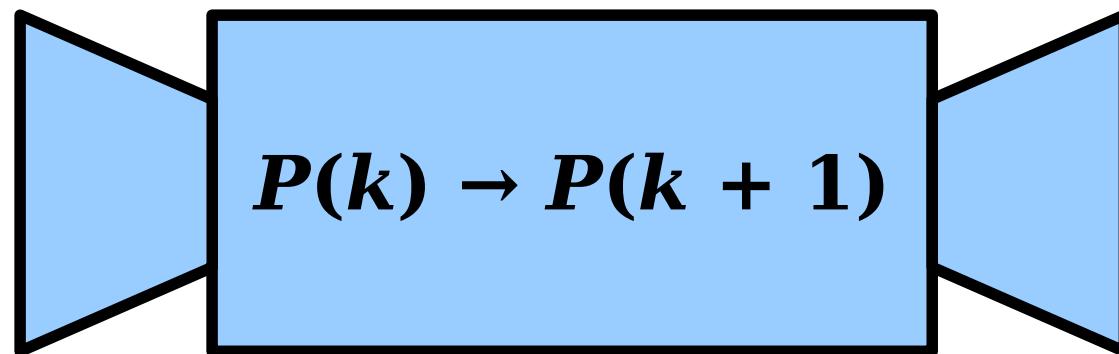
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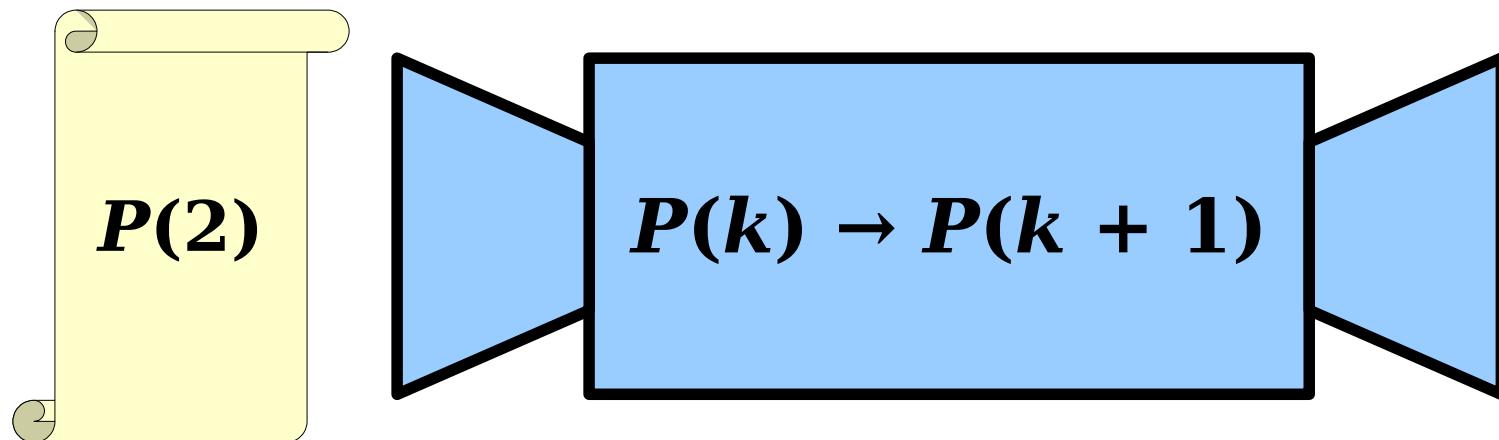
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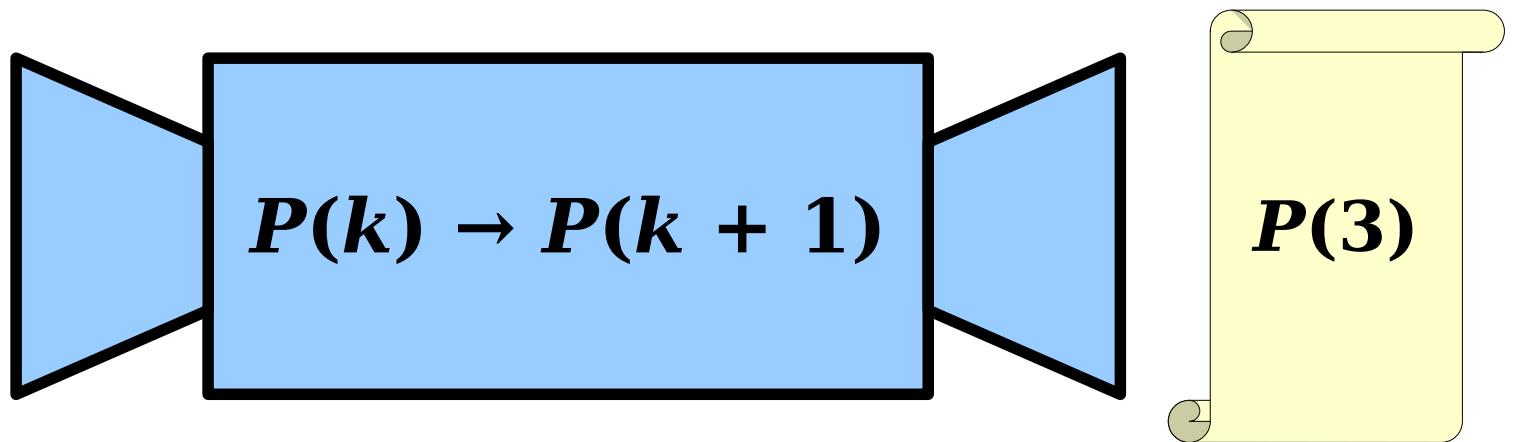
# Why Induction Works



# Why Induction Works



# Why Induction Works



# Proof by Induction

- A ***proof by induction*** is a way to use the principle of mathematical induction to show that some result is true for all natural numbers  $n$ .
- In a proof by induction, there are three steps:
  - Prove that  $P(0)$  is true.
    - This is called the ***basis*** or the ***base case***.
  - Prove that if  $P(k)$  is true, then  $P(k+1)$  is true.
    - This is called the ***inductive step***.
    - The assumption that  $P(k)$  is true is called the ***inductive hypothesis***.
  - Conclude, by induction, that  $P(n)$  is true for all  $n \in \mathbb{N}$ .

# Some Sums

**2<sup>0</sup>**

**2<sup>0</sup> + 2<sup>1</sup>**

**2<sup>0</sup> + 2<sup>1</sup> + 2<sup>2</sup>**

**2<sup>0</sup> + 2<sup>1</sup> + 2<sup>2</sup> + 2<sup>3</sup>**

**2<sup>0</sup> + 2<sup>1</sup> + 2<sup>2</sup> + 2<sup>3</sup> + 2<sup>4</sup>**

$$\mathbf{2^0} = 1$$

$$\mathbf{2^0 + 2^1} = 1 + 2 = 3$$

$$\mathbf{2^0 + 2^1 + 2^2} = 1 + 2 + 4 = 7$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3} = 1 + 2 + 4 + 8 = 15$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3 + 2^4} = 1 + 2 + 4 + 8 + 16 = 31$$

$$2^0 = 1 = 2^1 - 1$$

$$2^0 + 2^1 = 1 + 2 = 3 = 2^2 - 1$$

$$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7 = 2^3 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15 = 2^4 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31 = 2^5 - 1$$

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At the start of the proof, we tell the reader what predicate we're going to show is true for all natural numbers  $n$ , then tell them we're going to prove it by induction.

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Here, we state what  $P(0)$  actually says. Now, can go prove this using any proof techniques we'd like!

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The goal of this step is to prove

**“If  $P(k)$  is true, then  $P(k+1)$  is true.”**

So we ask the reader to pick some  $k$ , assume that  $P(k)$  is true, then try to prove  $P(k+1)$ .

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Here, we explicitly state  $P(k+1)$ , which is what we want to prove. Now, we can use any proof technique we want to prove it.

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Here, we'll use our **inductive hypothesis**

(the assumption that  $P(k)$  is true) to simplify a complex expression. This is a common theme in inductive proofs.

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We need to show that  $P(k + 1)$  holds, meaning that the sum of the first  $k + 1$  powers of two is  $2^{k+1} - 1$ . To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \dots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \dots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \quad (\text{via (1)}) \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

**Theorem:** The sum of the first  $n$  powers of two is  $2^n - 1$ .

**Proof:** Let  $P(n)$  be the statement “the sum of the first  $n$  powers of two is  $2^n - 1$ .” We will prove, by induction, that  $P(n)$  is true for all  $n \in \mathbb{N}$ , from which the theorem follows.

For our base case, we need to show  $P(0)$  is true, meaning that the sum of the first zero powers of two is  $2^0 - 1$ . Since the sum of the first zero powers of two is zero and  $2^0 - 1$  is zero as well, we see that  $P(0)$  is true.

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Therefore,  $P(k + 1)$  is true, completing the induction. ■

# A Quick Aside

- This result helps explain the range of numbers that can be stored in an `int`.
- If you have an unsigned 32-bit integer, the largest value you can store is given by  $1 + 2 + 4 + 8 + \dots + 2^{31} = 2^{32} - 1$ .
- This formula for sums of powers of two has many other uses as well. You'll see one on Friday.

# Structuring a Proof by Induction

- Define some predicate  $P$  that you'll show, by induction, is true for all natural numbers.
- Prove the base case:
  - State that you're going to prove that  $P(0)$  is true, then go prove it.
- Prove the inductive step:
  - Say that you're assuming  $P(k)$  for some arbitrary natural number  $k$ , then write out exactly what that means.
  - Say that you're going to prove  $P(k+1)$ , then write out exactly what that means.
  - Prove that  $P(k+1)$  using any proof technique you'd like!
- This is a rather verbose way of writing inductive proofs. As we get more experience with induction, we'll start leaving out some details from our proofs.

# The Counterfeit Coin Problem

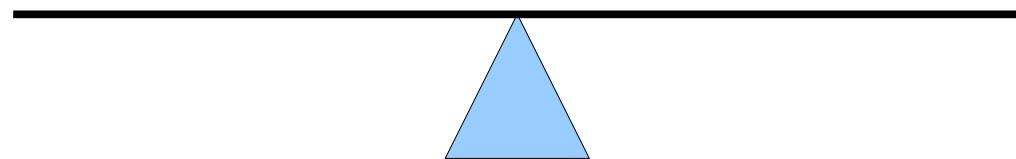
# Problem Statement

- You are given a set of three seemingly identical coins, two of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only one weighing on the balance, find the counterfeit coin.

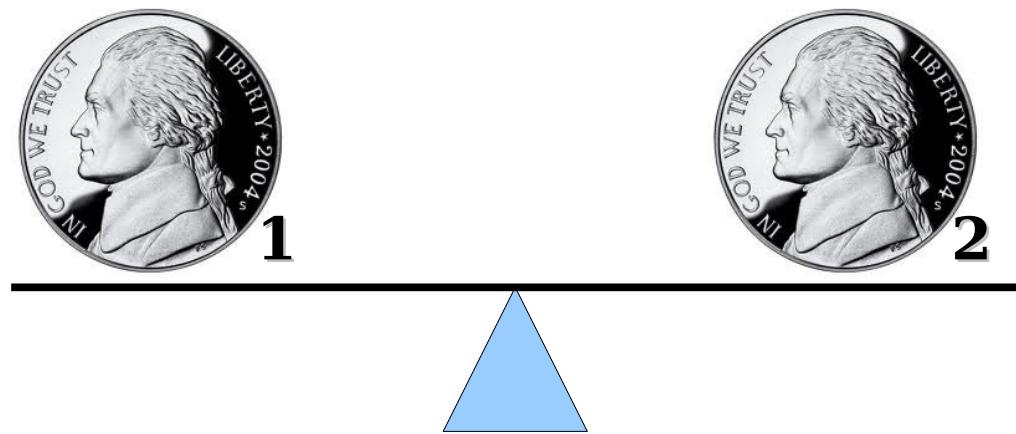
How?

Answer at  
<https://cs103.stanford.edu/pollev>

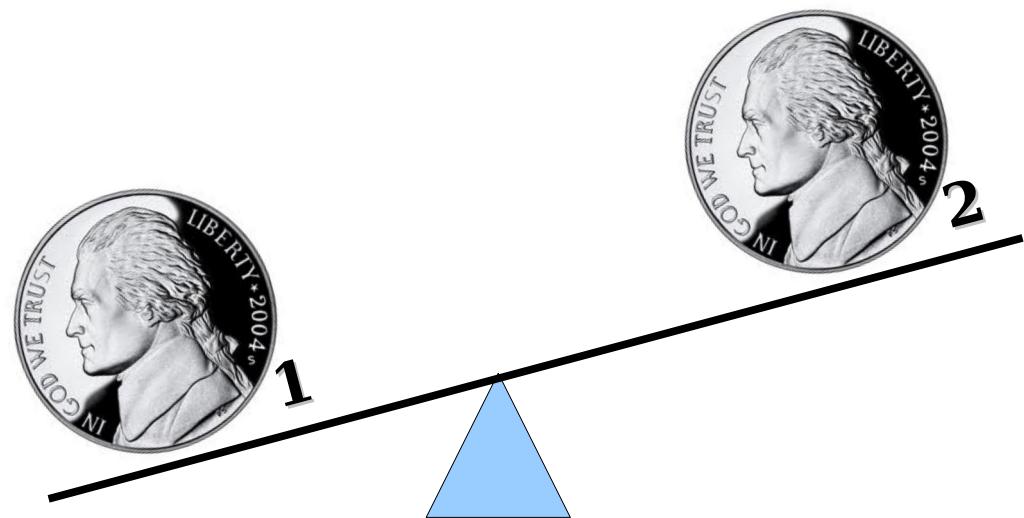
# Finding the Counterfeit Coin



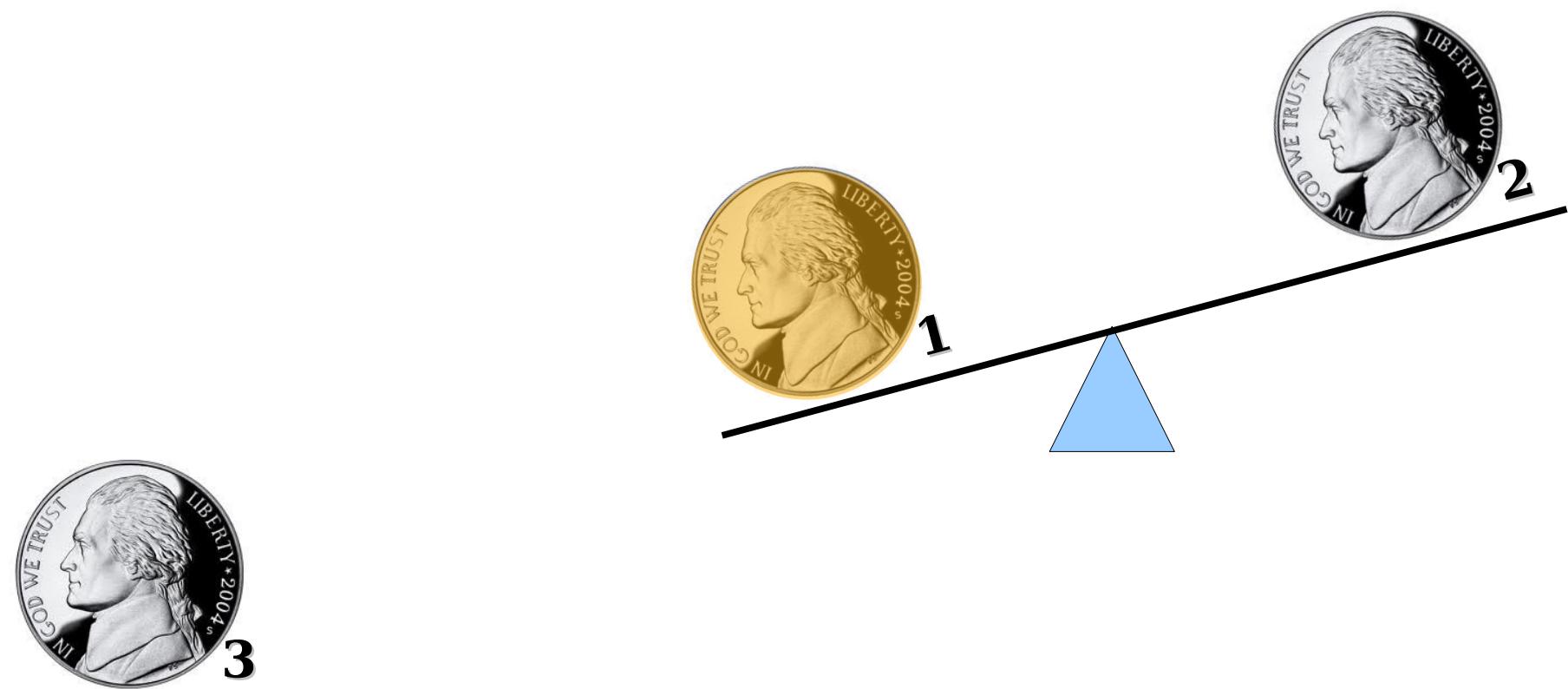
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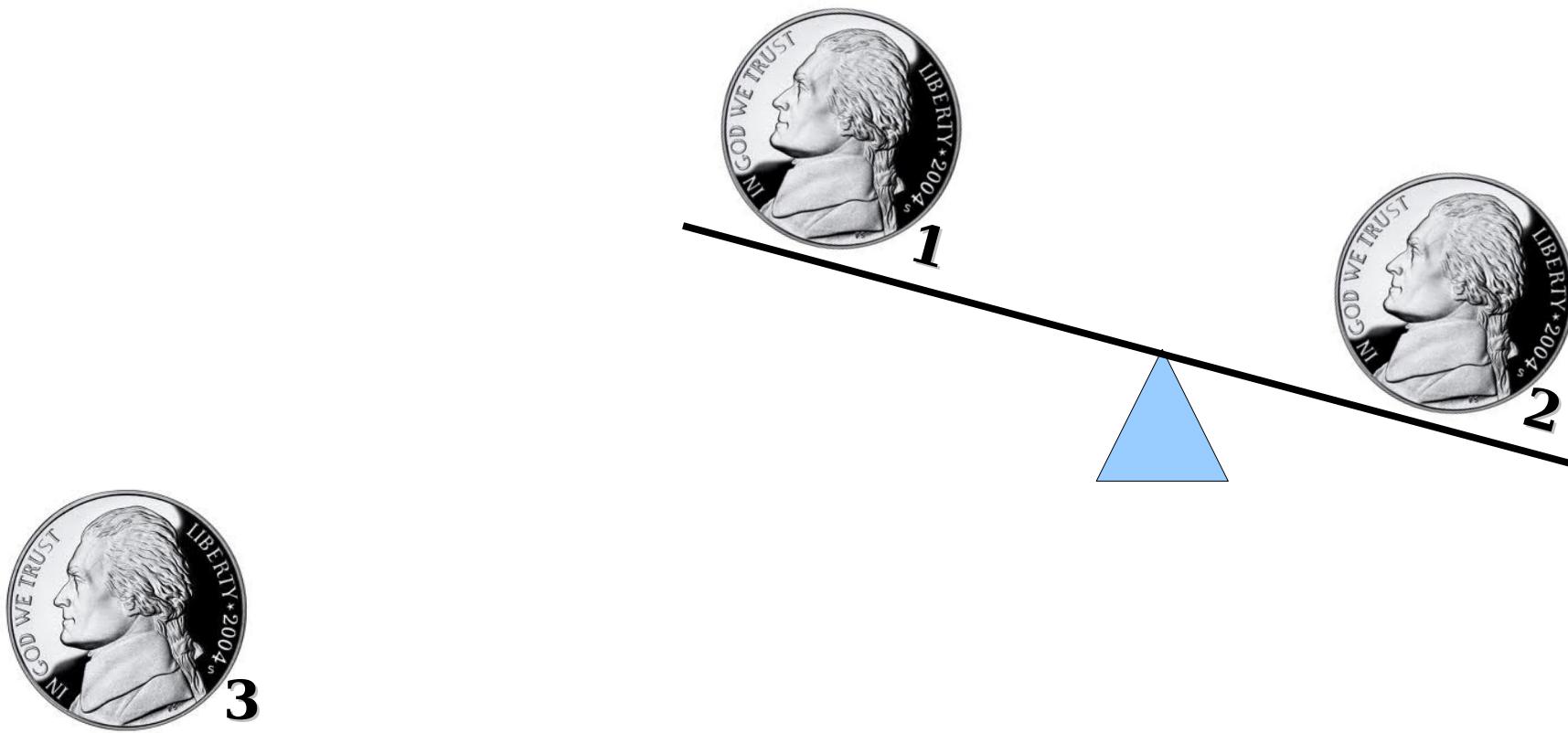
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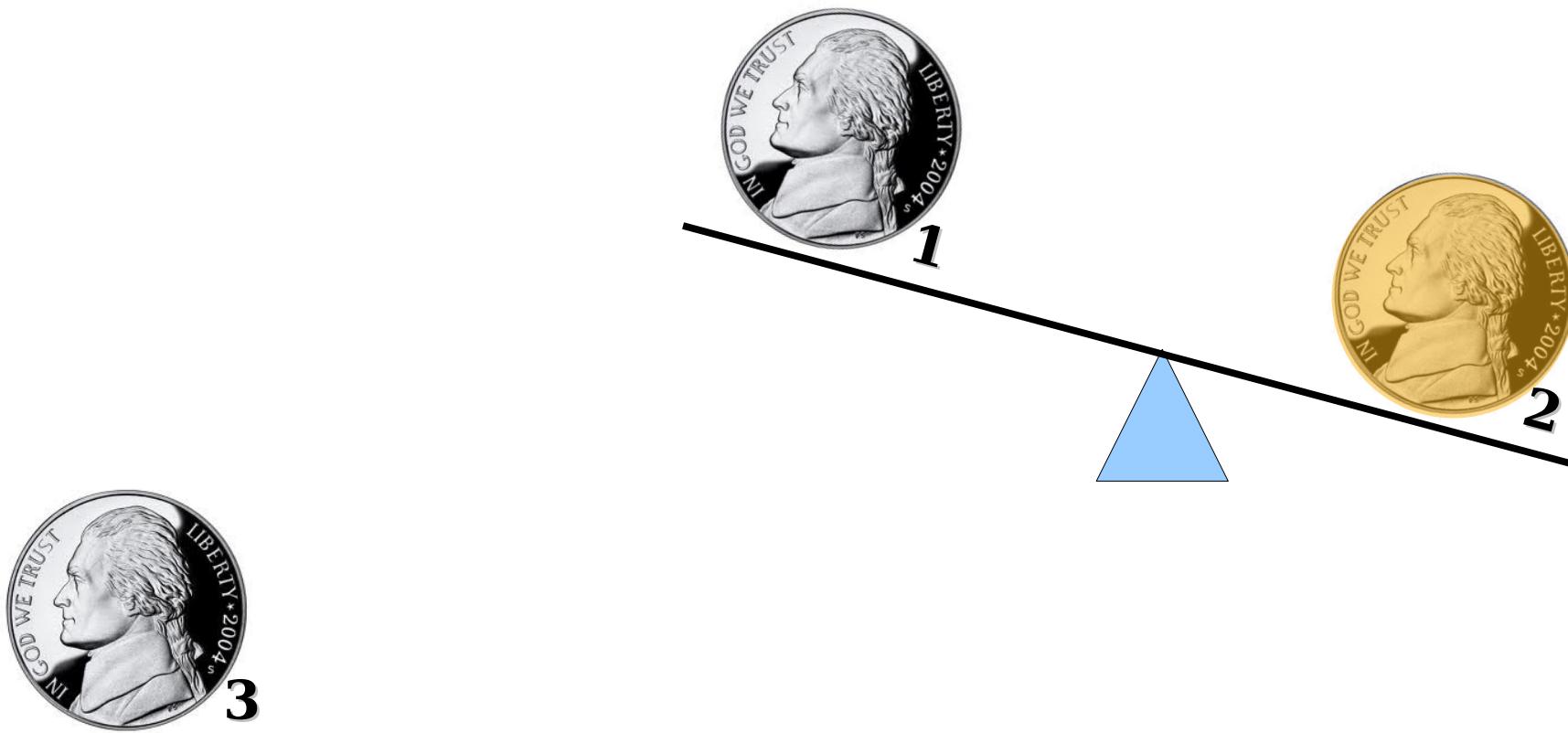
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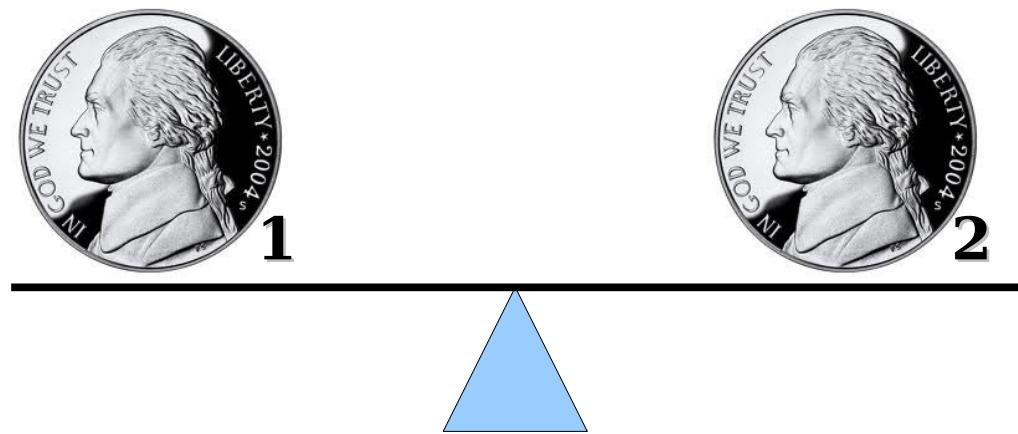
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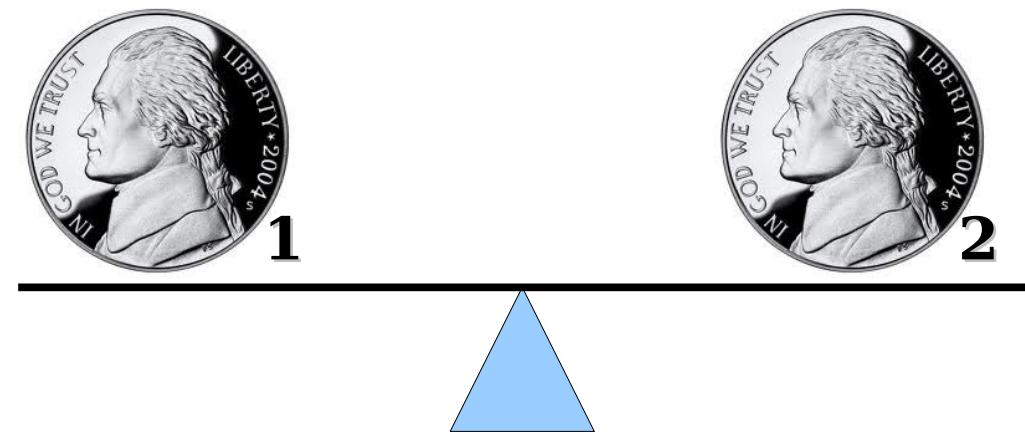
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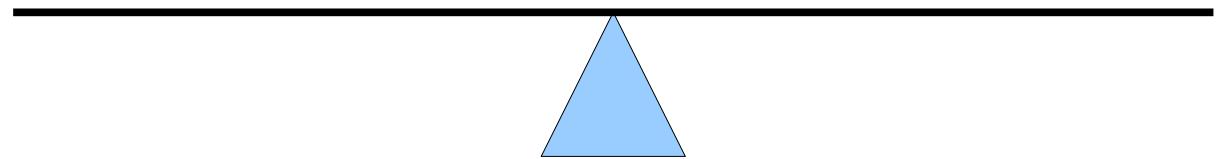
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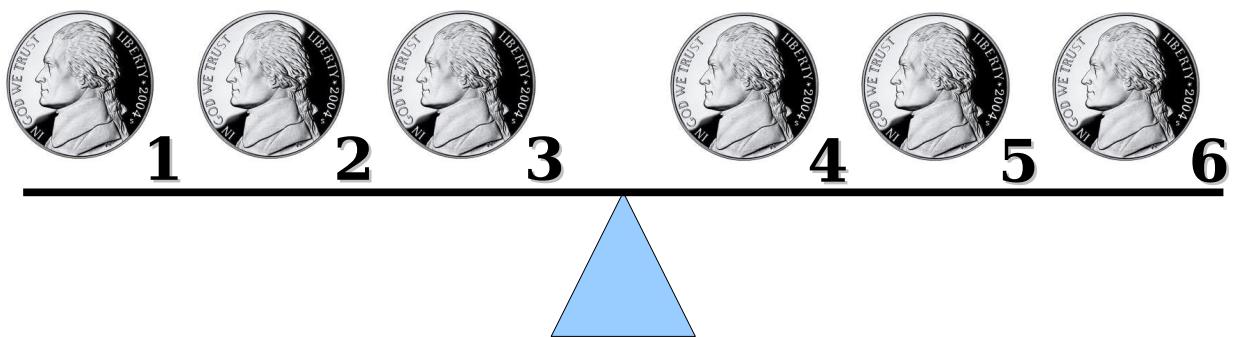
# A Harder Problem

- You are given a set of **nine** seemingly identical coins, eight of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only **two** weighings on the balance, find the counterfeit coin.

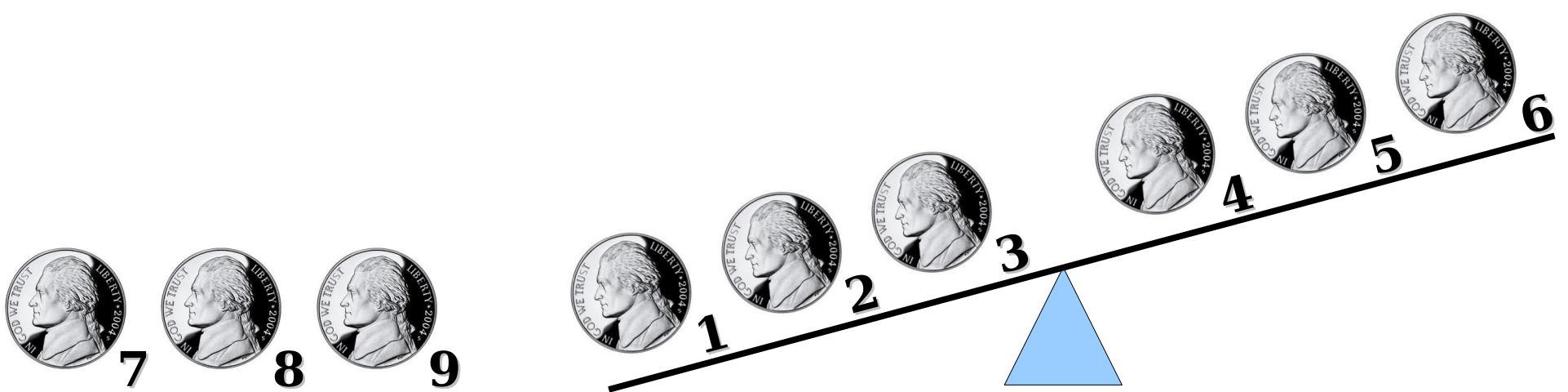
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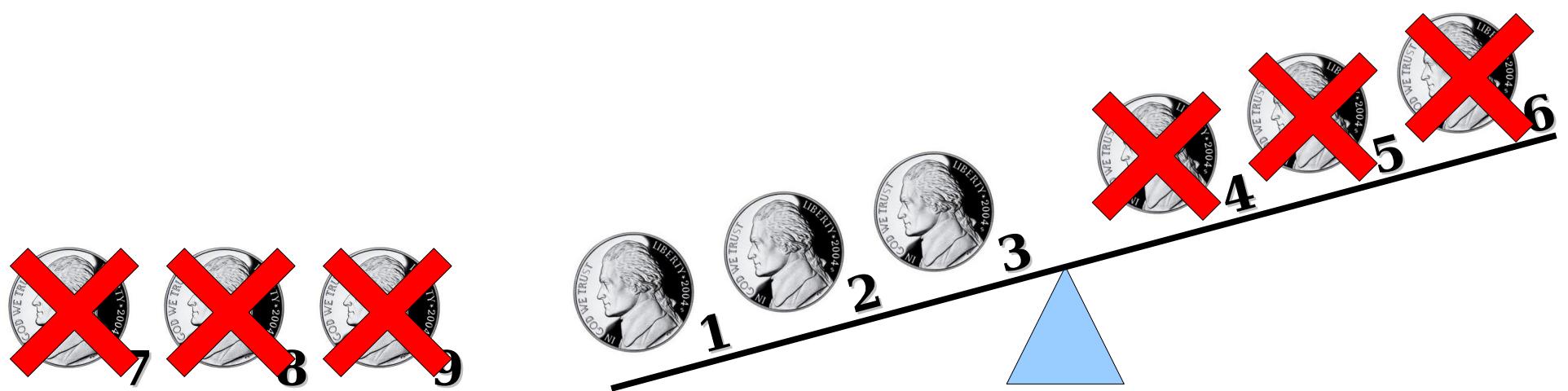
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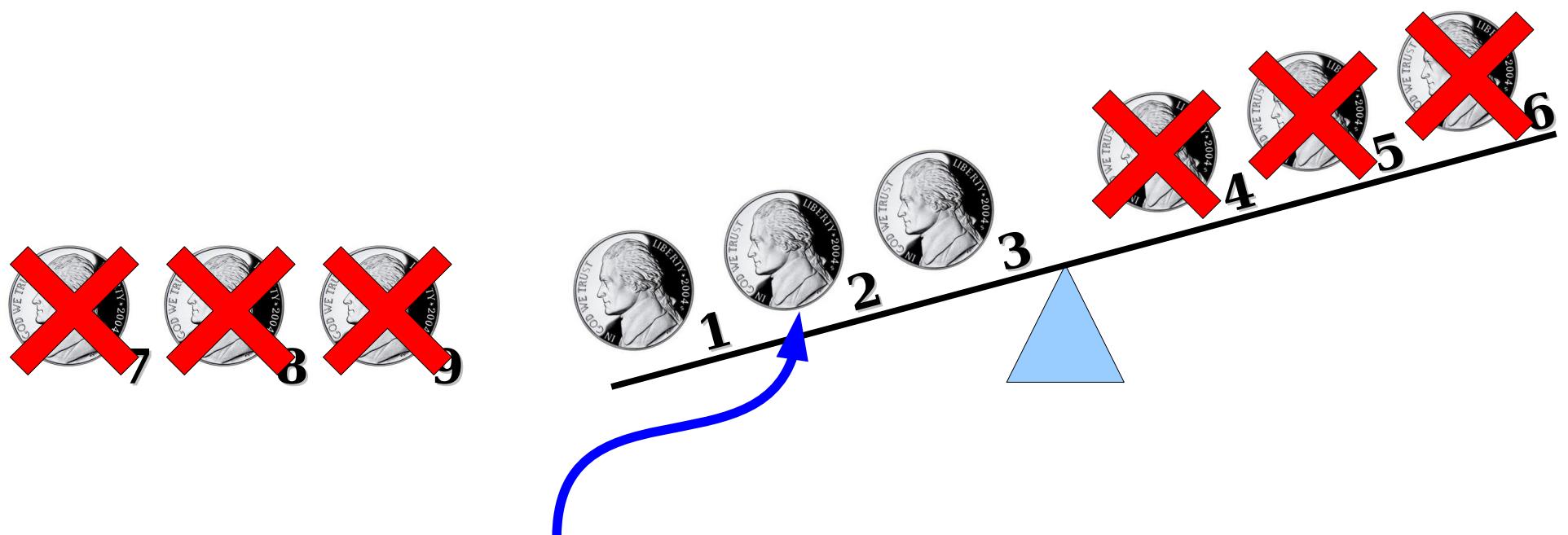
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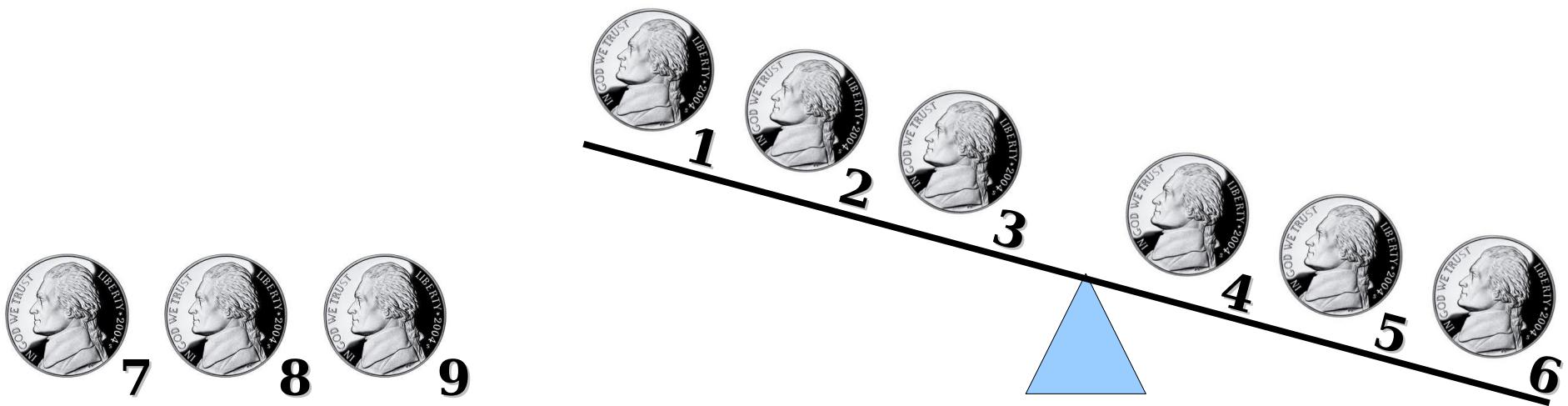


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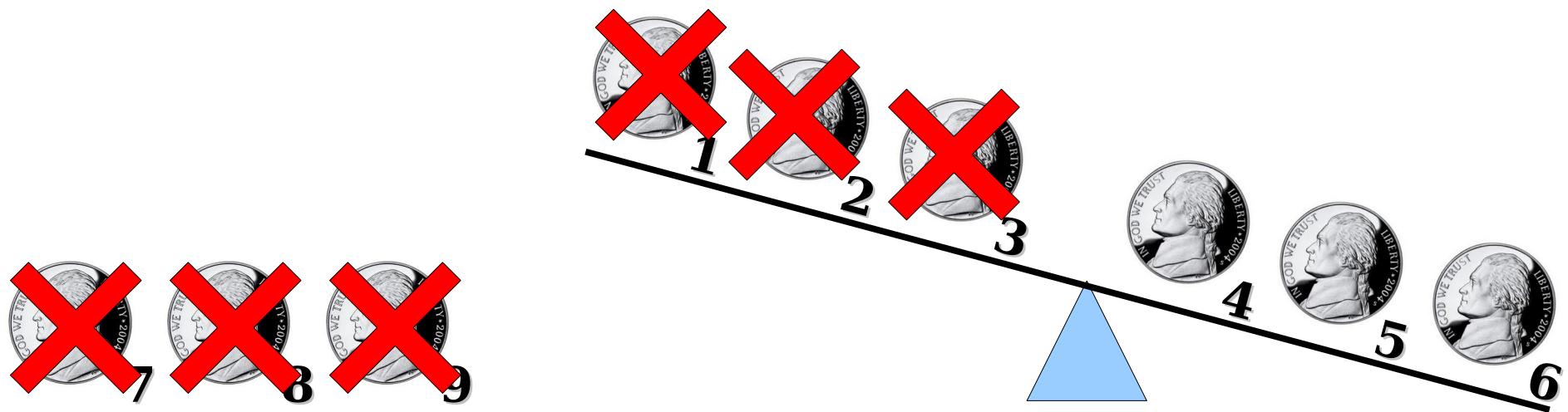


Now we have one weighing  
to find the counterfeit out  
of these three coins.

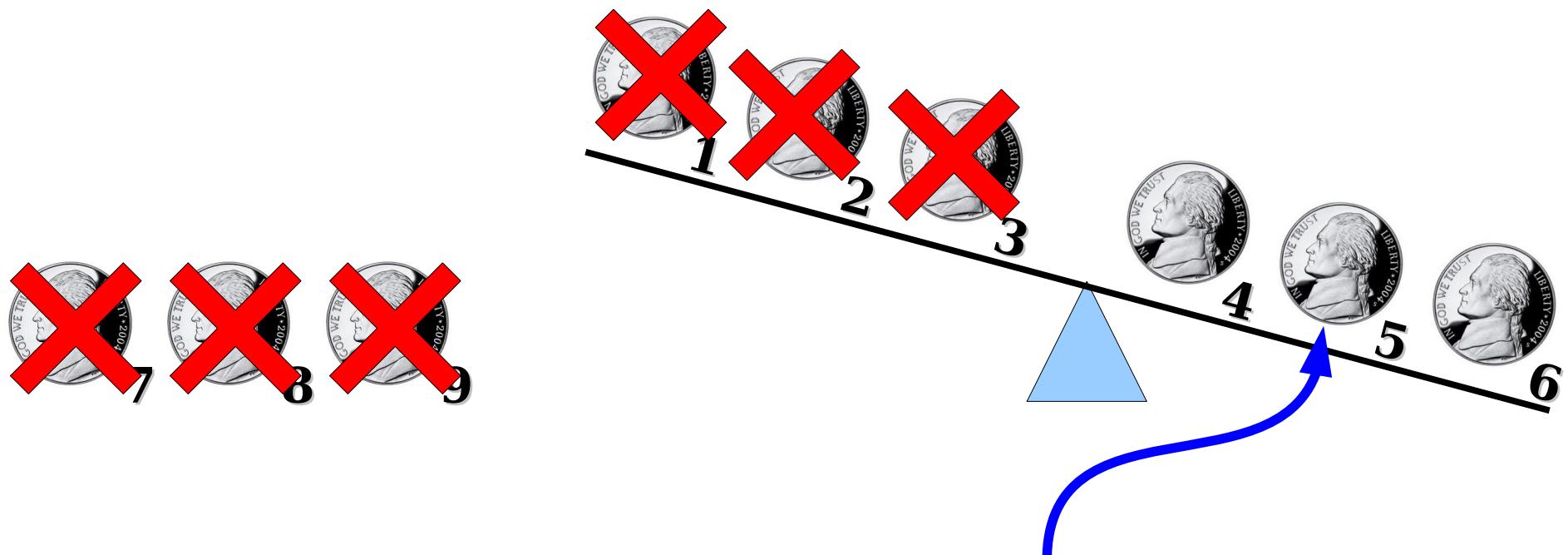
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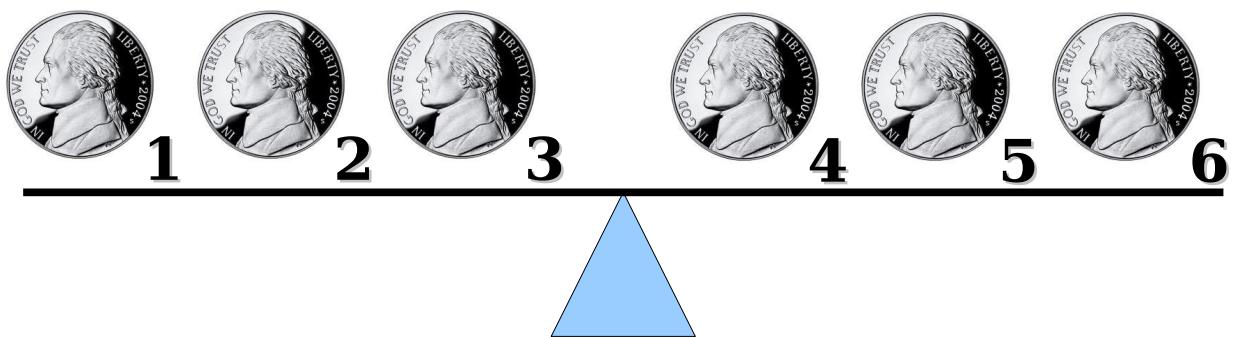


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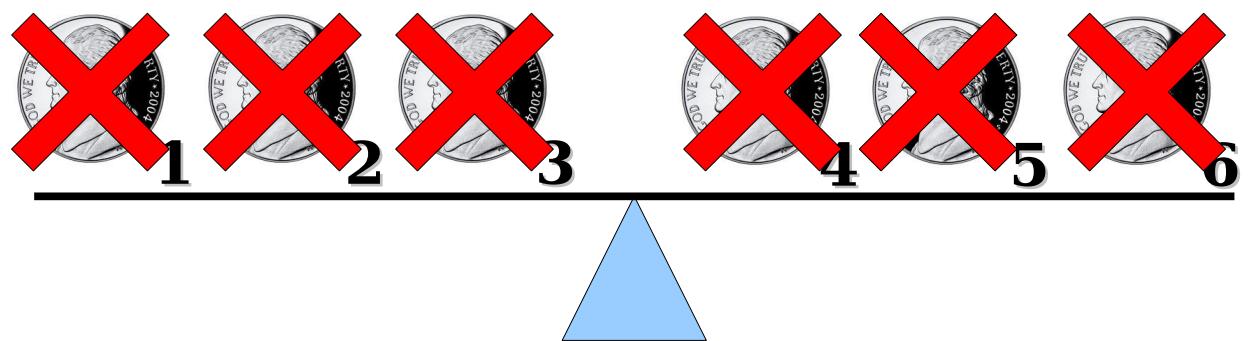


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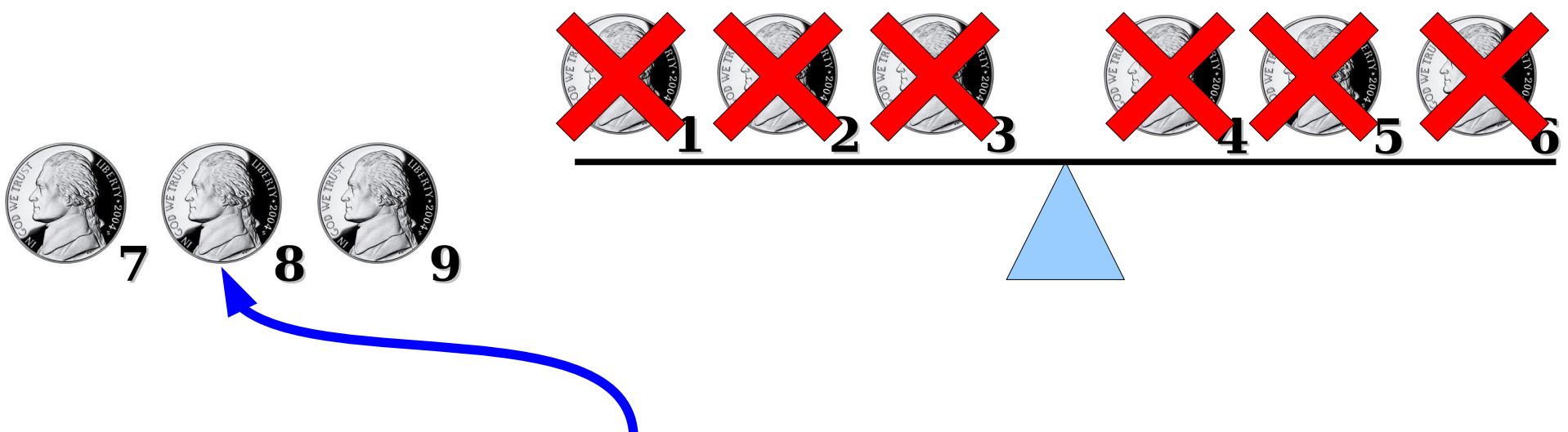
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# Finding the Counterfeit Coin



Now we have one weighing  
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of these three coins.

Can we generalize this?

# A Pattern

- Assume out of the coins that are given, exactly one is counterfeit and weighs more than the other coins.
- If we have no weighings, how many coins can we have while still being able to find the counterfeit?
  - **One** coin, since that coin has to be the counterfeit!
- If we have one weighing, we can find the counterfeit out of **three** coins.
- If we have two weighings, we can find the counterfeit out of **nine** coins.

So far, we have

$$1, 3, 9 = 3^0, 3^1, 3^2$$

Does this pattern continue?

**Theorem:** If exactly one coin in a group of  $3^n$  coins is heavier than the rest, that coin can be found using only  $n$  weighings on a balance.

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At the start of the proof, we tell the reader what predicate we're going to show is true for all natural numbers  $n$ , then tell them we're going to prove it by induction.

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The goal of this step is to prove

**“If  $P(k)$  is true, then  $P(k+1)$  is true.”**

So we ask the reader to choose an arbitrary  $k$ , assume that  $P(k)$  is true, then try to prove  $P(k+1)$ .

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We'll use induction to prove that  $P(n)$  holds for every  $n \in \mathbb{N}$ , from which the theorem follows.

As our base case, we'll prove that  $P(0)$  is true, meaning that if we have a set of  $3^0=1$  coins with one coin heavier than the rest, we can find that coin with zero weighings. This is true because if we have just one coin, it's vacuously heavier than all the others, and no weighings are needed.

For the inductive step, suppose  $P(k)$  is true for some arbitrary  $k \in \mathbb{N}$ , so we can find the heavier of  $3^k$  coins in  $k$  weighings. **We'll prove  $P(k+1)$ :** that we can find the heavier of  $3^{k+1}$  coins in  $k+1$  weighings.

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Here, we explicitly state  $P(k+1)$ , which is what we want to prove. Now, we can use any proof technique we want to try to prove it.

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We'll use induction to prove this statement. By the theorem,  $P(1)$  is true.

As our base case, consider a set of  $3^0$  coins.

With a set of one coin, we have zero weighings. We find that

Here, we use our **inductive hypothesis** (the assumption that  $P(k)$  is true) to solve this simpler version of the overall problem.

For the inductive step, suppose  $P(k)$  is true for some arbitrary  $k \in \mathbb{N}$ , so we can find the heavier of  $3^k$  coins in  $k$  weighings. We'll prove  $P(k+1)$ : that we can find the heavier of  $3^{k+1}$  coins in  $k+1$  weighings.

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As our base case, we'll prove that  $P(0)$  is true, meaning that if we have a set of  ~~$3^0 - 1$  coins with one coin heavier than the rest, we can find that coin with one coin heavier than the rest.~~ In a proof by induction, we need to prove that it's vacuous.

For the base case, we can see that we can find the heavy coin in a group of 1 coin, so  $P(0)$  is true. Now, we need to prove that if  $P(k)$  is true, then  $P(k+1)$  is true.

Suppose we have a group of  $3^{k+1}$  coins. We divide the group into three groups of  $3^k$  coins each. Weigh two of the groups against one another. If one group is heavier than the other, the coins in that group must contain the heavier coin. Otherwise, the heavier coin must be in the group we didn't put on the scale. Therefore, with one weighing, we can find a group of  $3^k$  coins containing the heavy coin. We can then use  $k$  more weighings to find the heavy coin in that group.

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In a proof by induction, we need to prove that  $P(0)$  is true, and that if  $P(k)$  is true, then  $P(k+1)$  is true.

For the base case, we can find the heavy coin in a set of one coin, so  $P(0)$  is true. For the inductive step, we assume  $P(k)$  is true, and we want to show  $P(k+1)$  is true. We do this by dividing a set of  $3^{k+1}$  coins into three groups of  $3^k$  coins each. Weigh two of the groups against one another. If one group is heavier than the other, the coins in that group must contain the heavier coin. Otherwise, the heavier coin must be in the group we didn't put on the scale. Therefore, with one weighing, we can find a group of  $3^k$  coins containing the heavy coin. We can then use  $k$  more weighings to find the heavy coin in that group.

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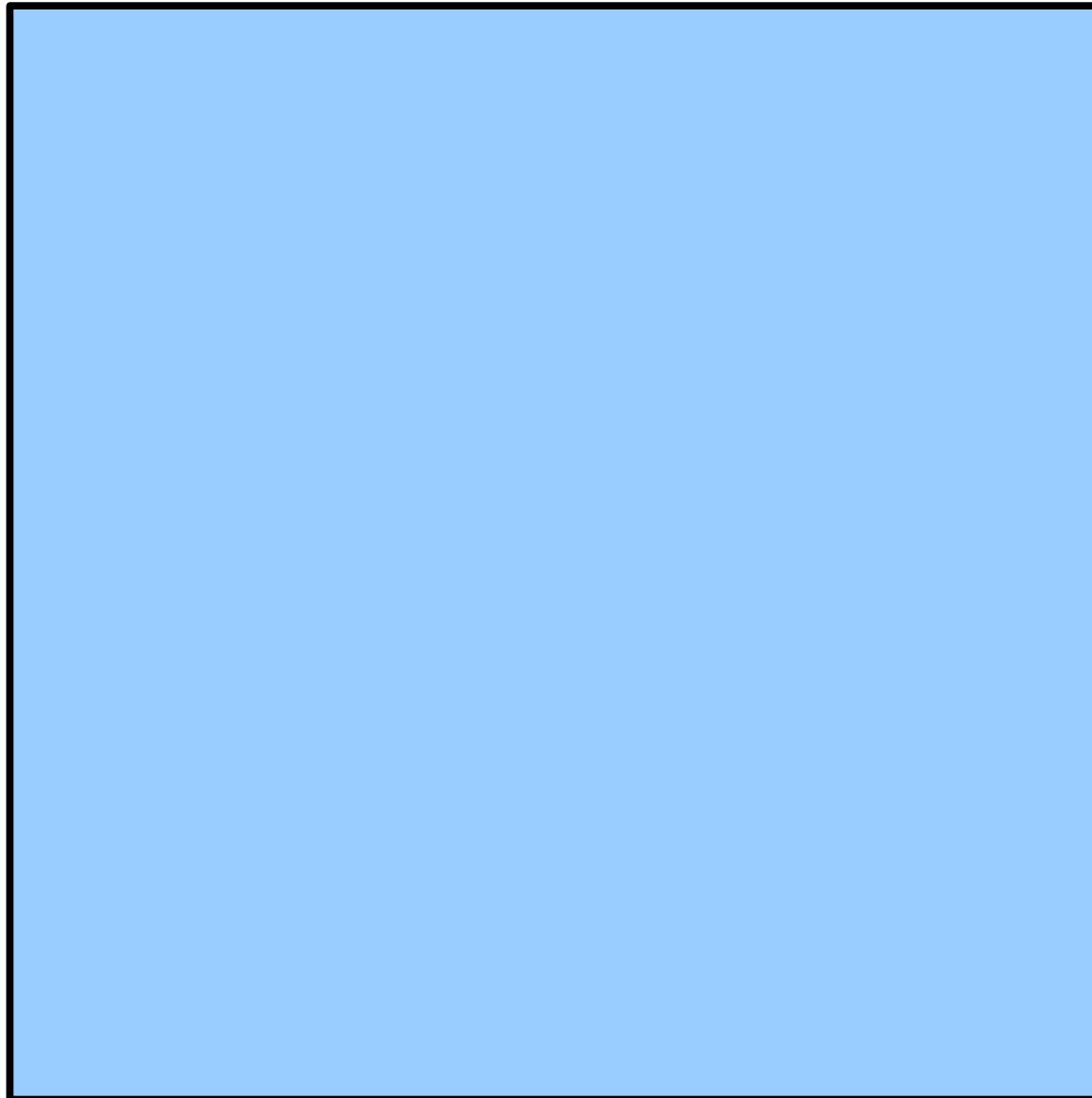
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# Some Fun Problems

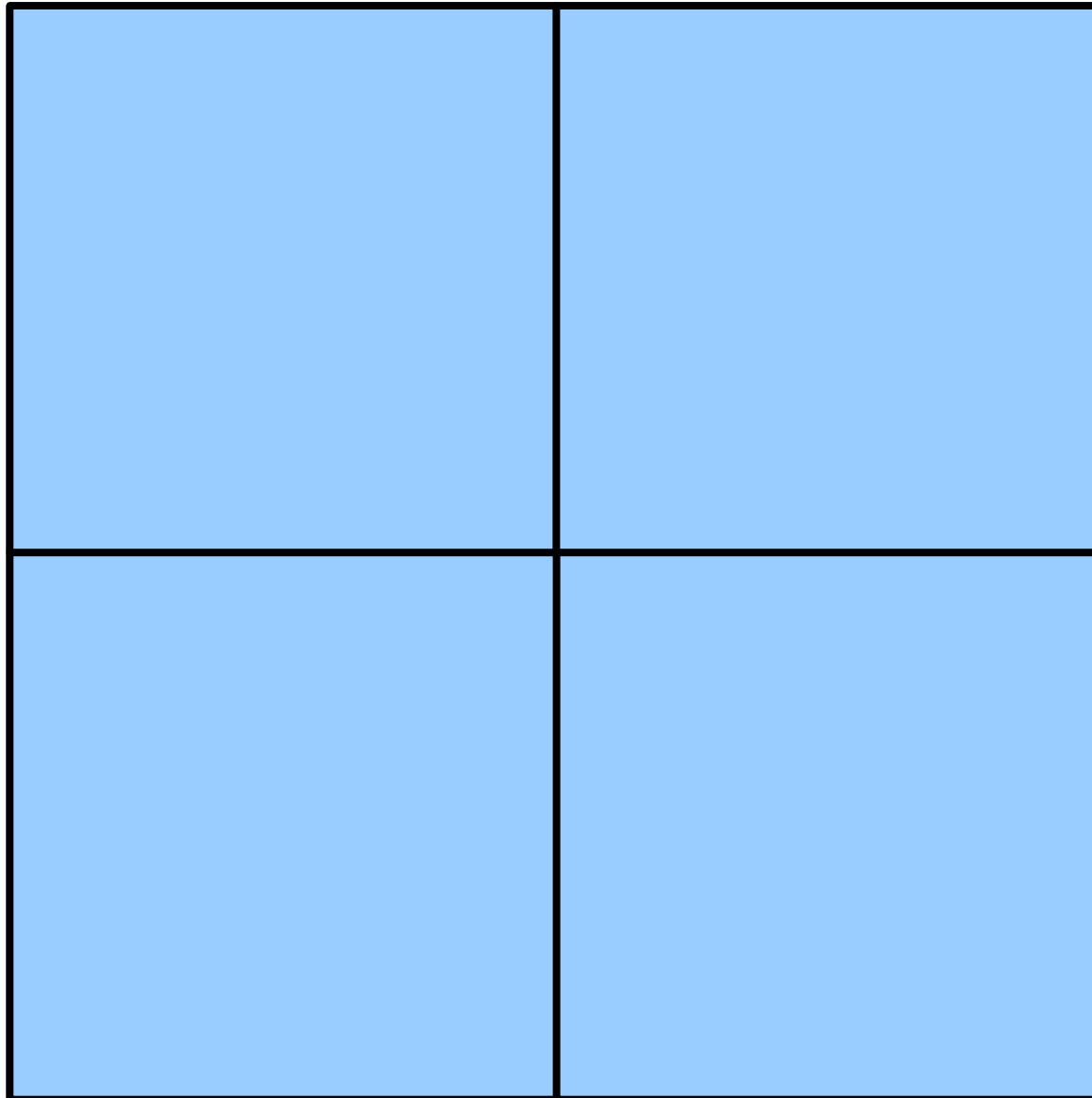
- Here's some nifty variants of this problem that you can work through:
  - Suppose that you have a group of coins where there's either exactly one heavier coin, or all coins weigh the same amount. If you only get  $k$  weighings, what's the largest number of coins where you can find the counterfeit or determine none exists?
  - What happens if the counterfeit can be either heavier or lighter than the other coins? What's the maximum number of coins where you can find the counterfeit if you have  $k$  weighings?
  - Can you find the counterfeit out of a group of more than  $3^k$  coins with  $k$  weighings?
  - Can you find the counterfeit out of any group of at most  $3^k$  coins with  $k$  weighings?

# Variations on Induction

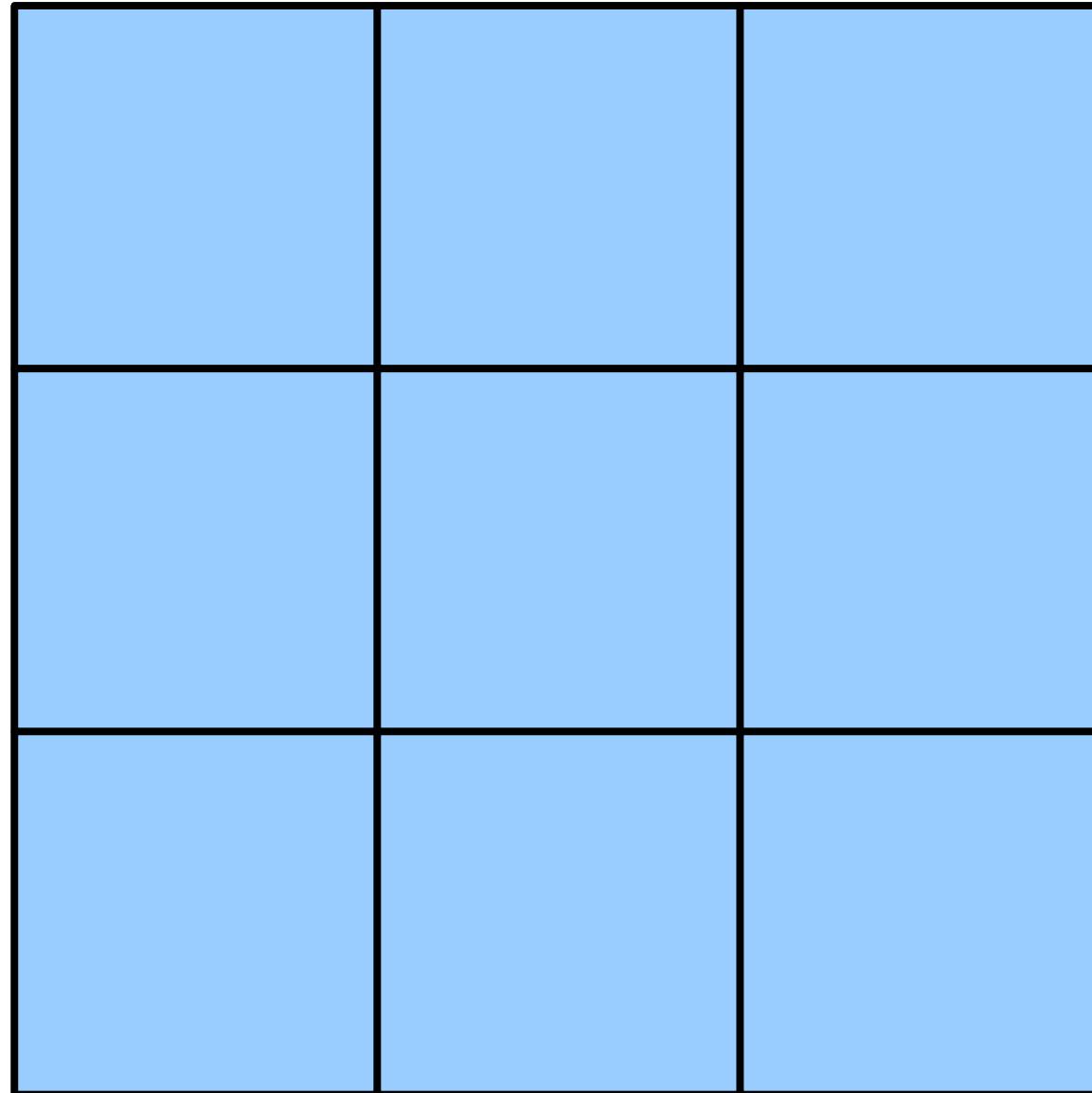
# Subdividing a Square



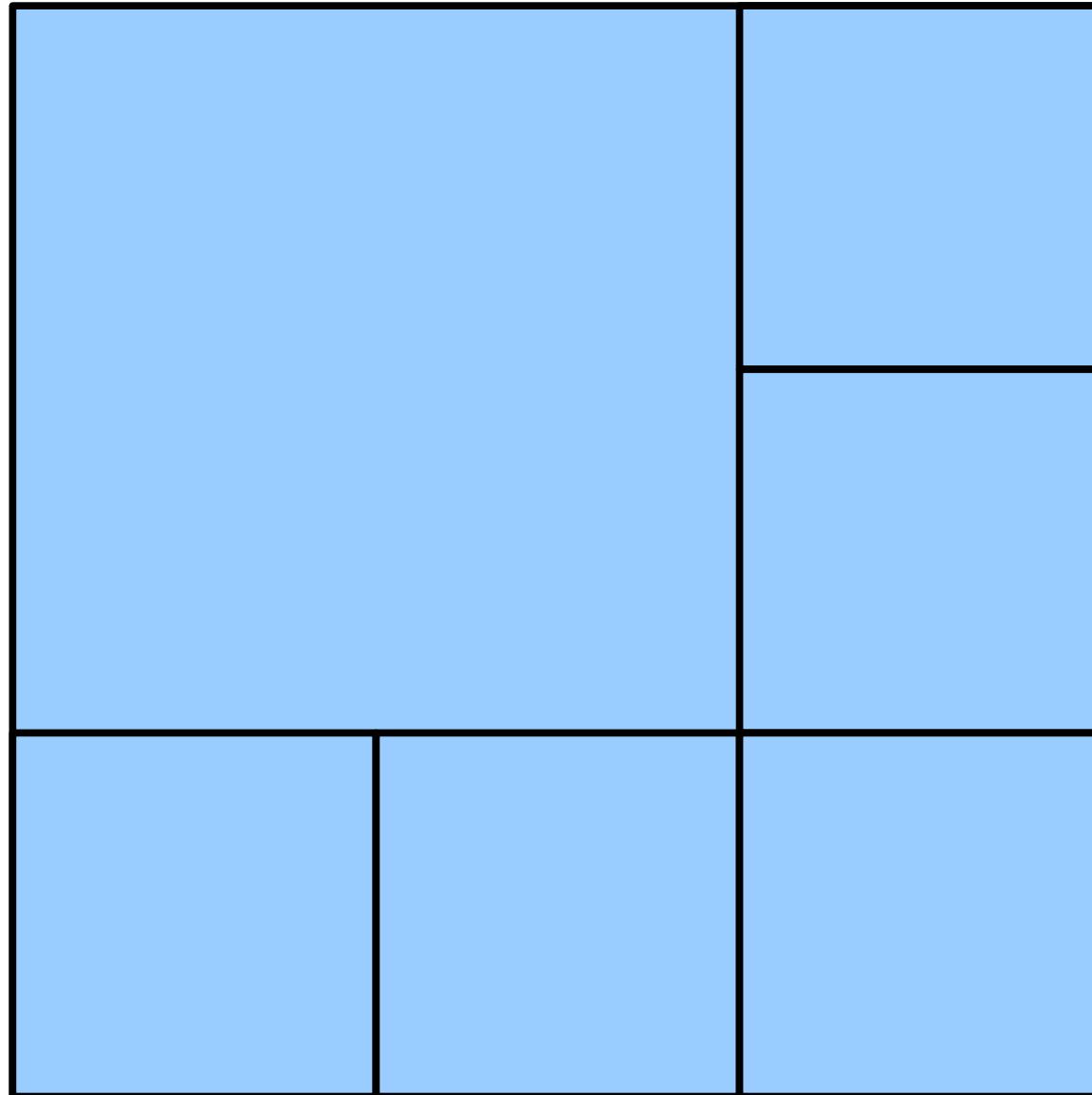
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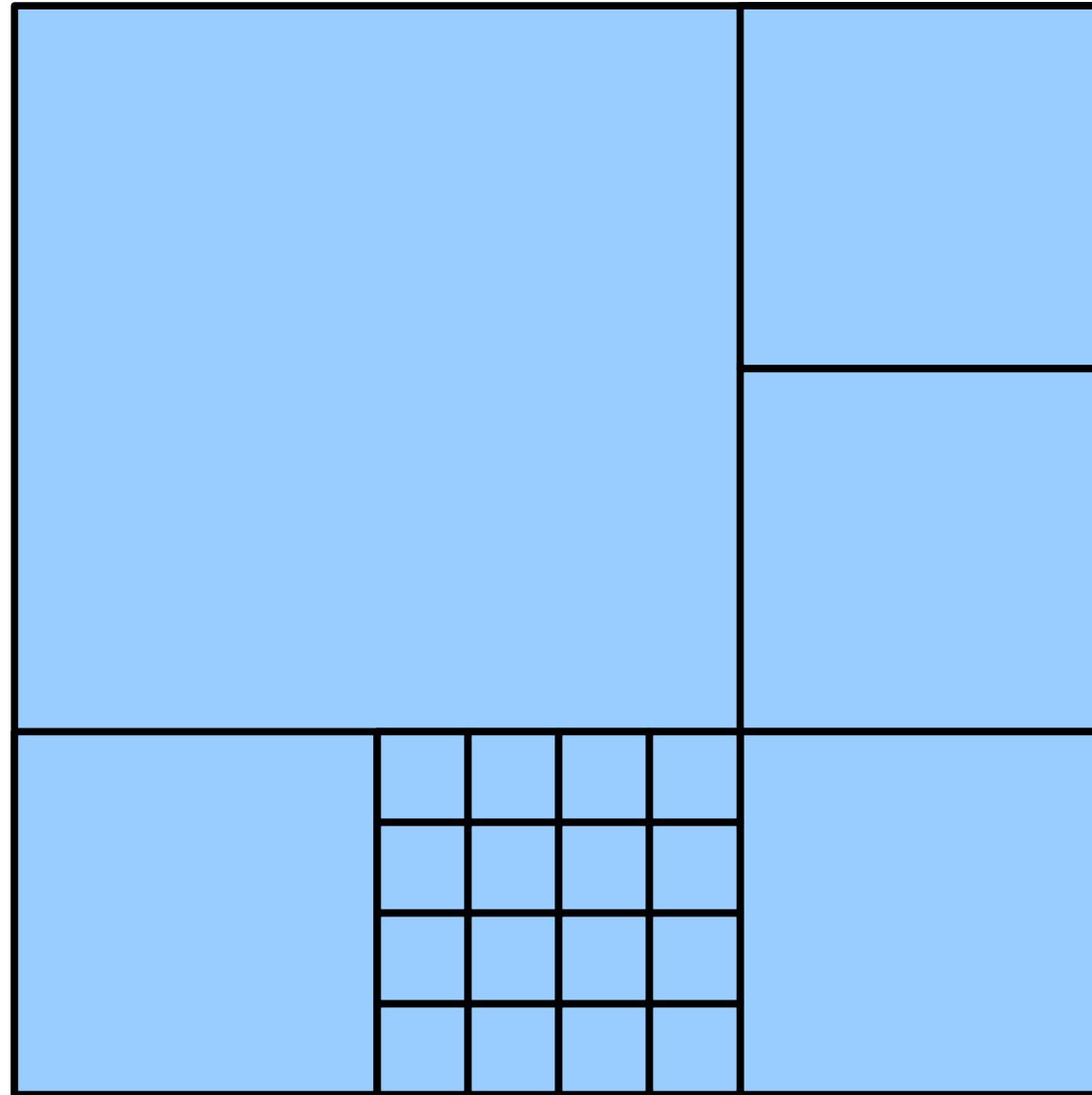
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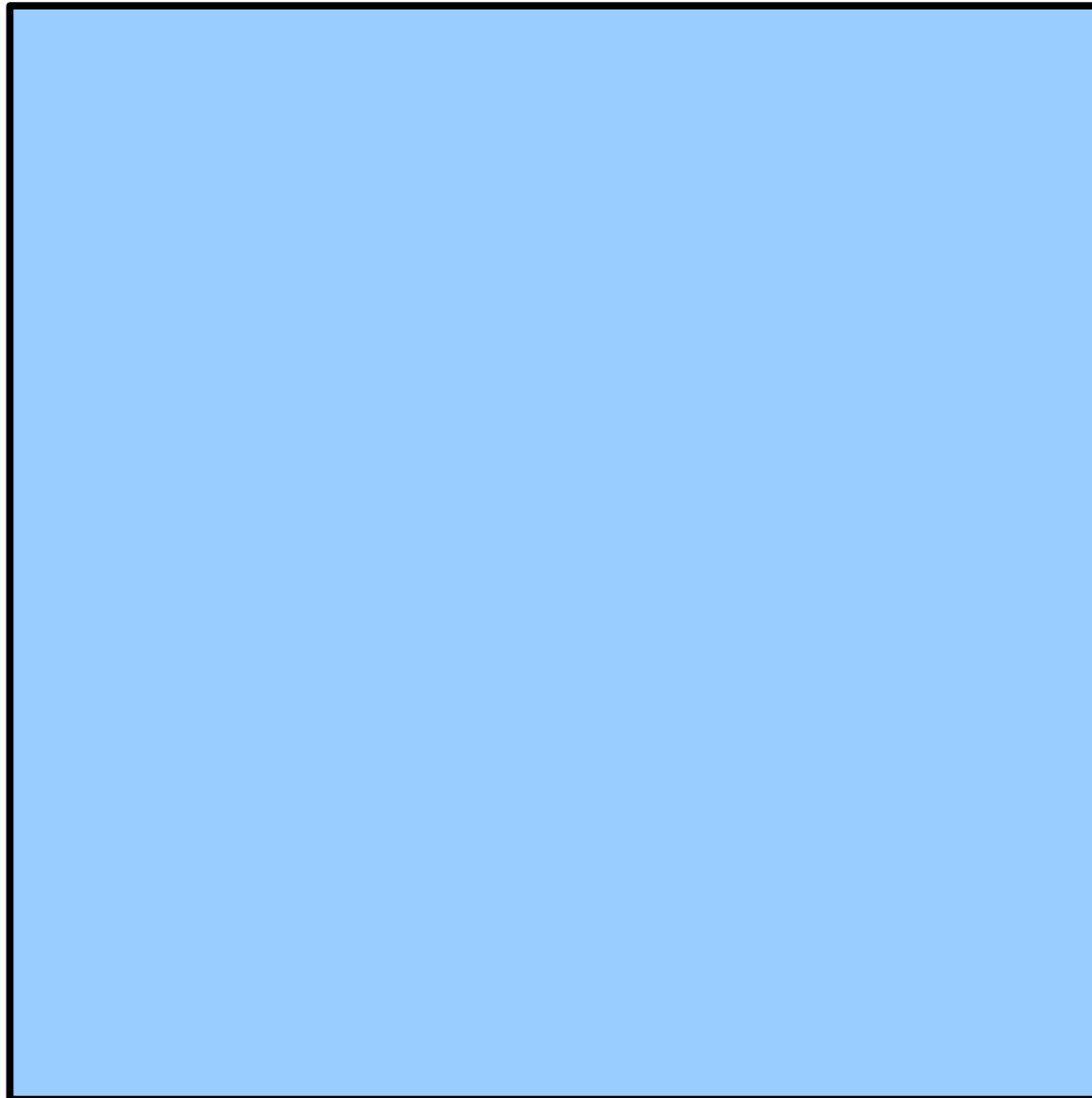
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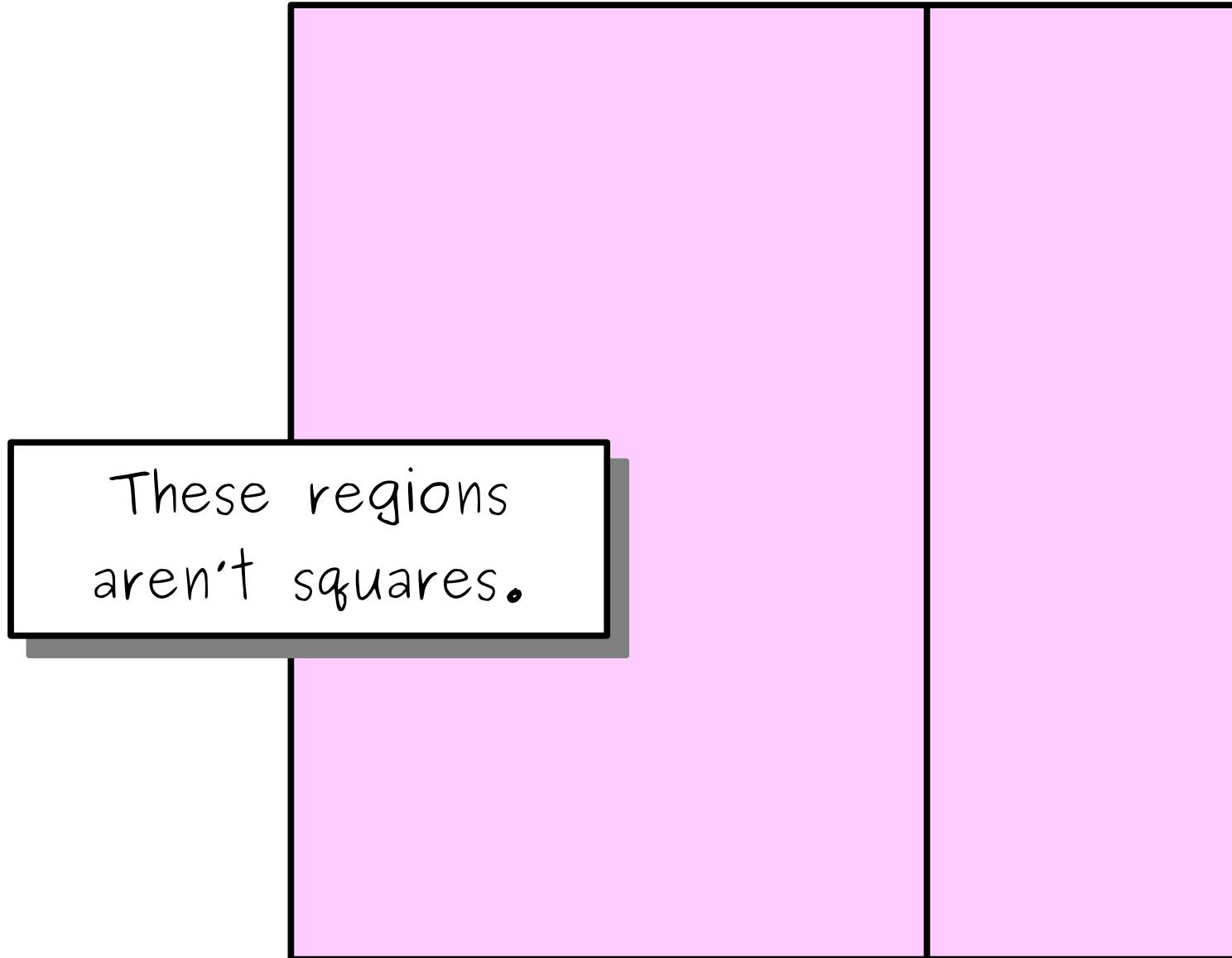
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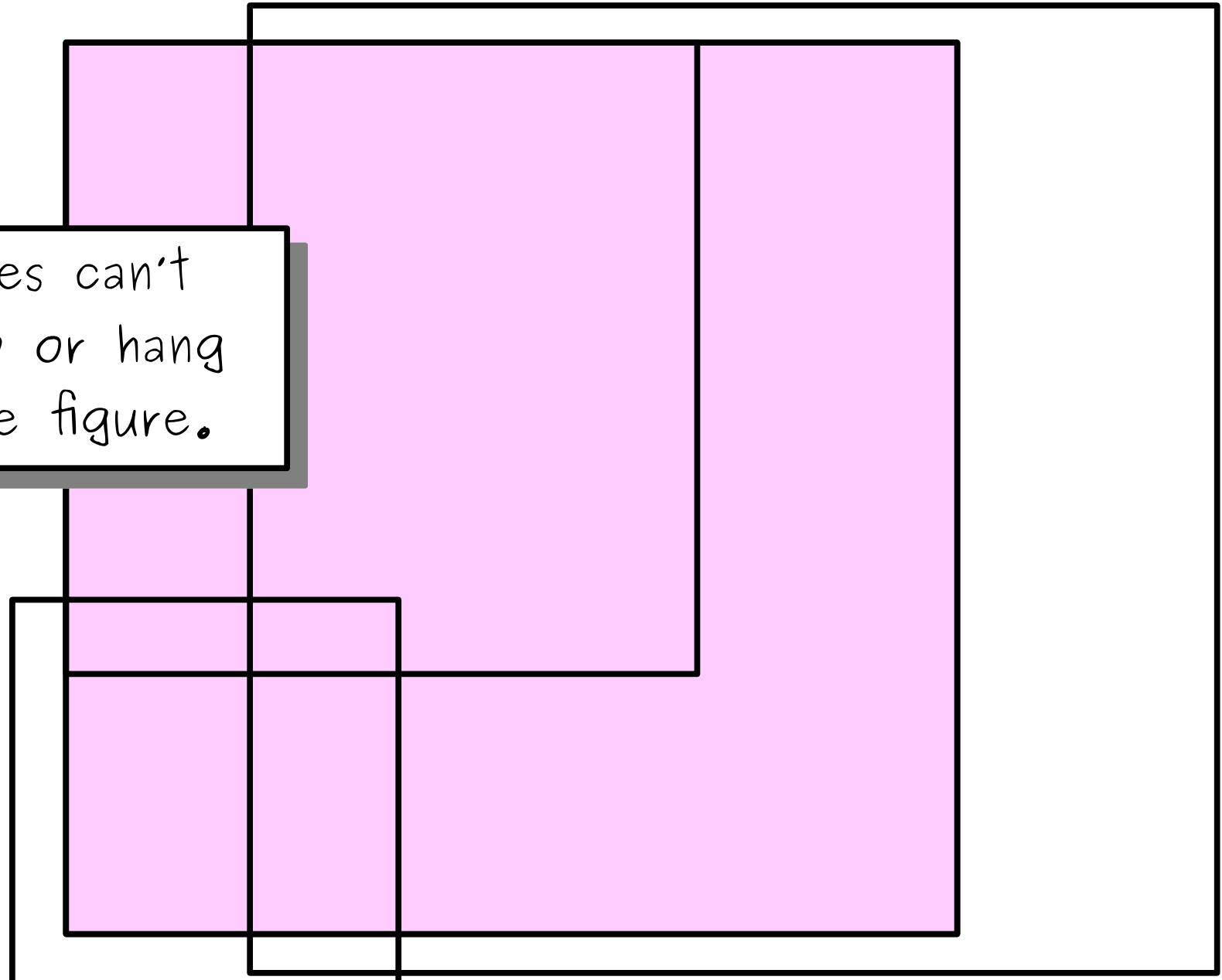


# Subdividing a Square



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Squares can't overlap or hang off the figure.



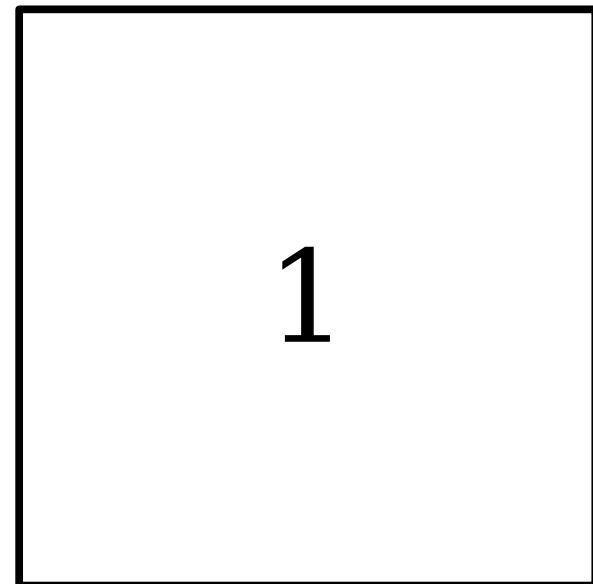
For what values of  $n$  can a square be subdivided into  $n$  squares?

Try out some numbers  $n$  from 1 to 12. Which values of  $n$  work?

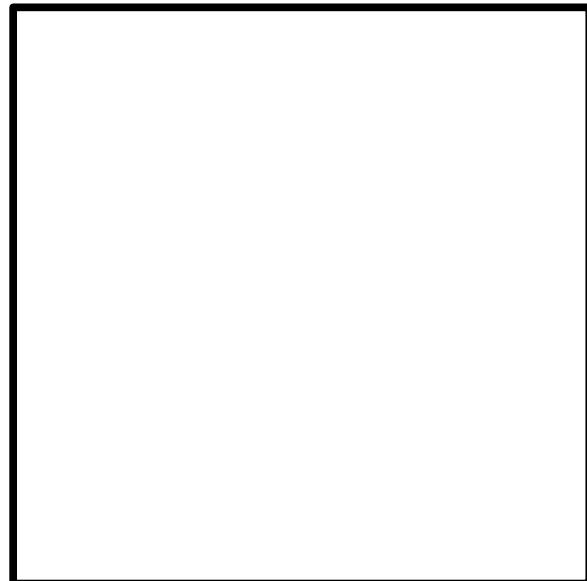
Answer at  
<https://cs103.stanford.edu/pollev>

1 2 3 4 5 6 7 8 9 10 11 12

1 2 3 4 5 6 7 8 9 10 11 12



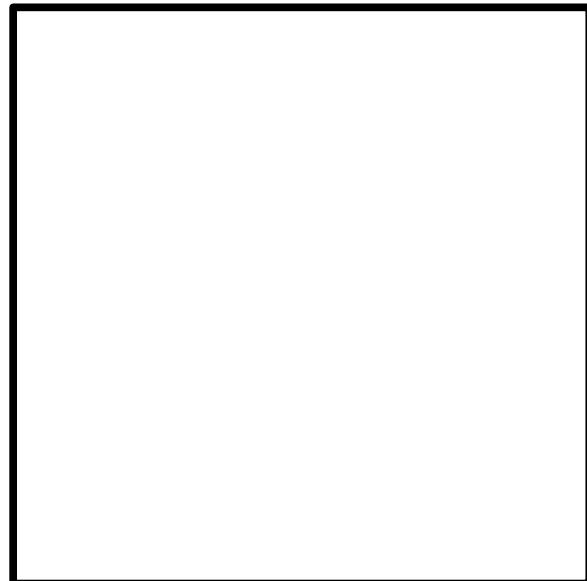
1 **2** **3** 4 5 6 7 8 9 10 11 12



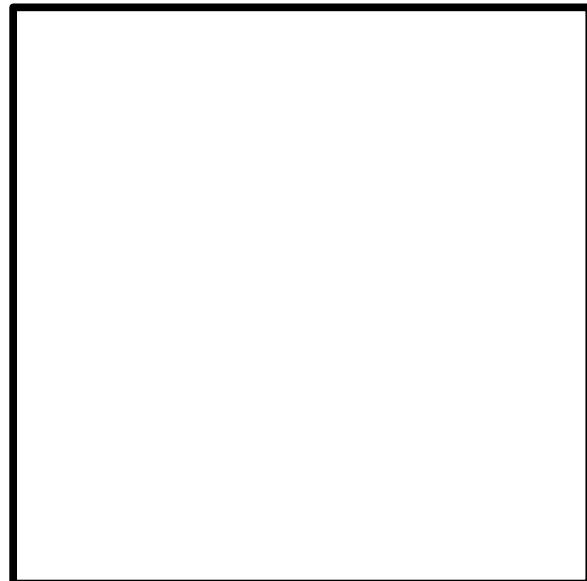
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1	2
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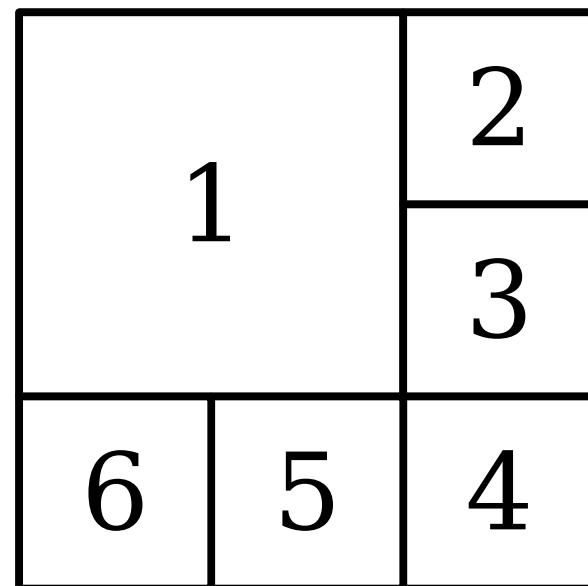
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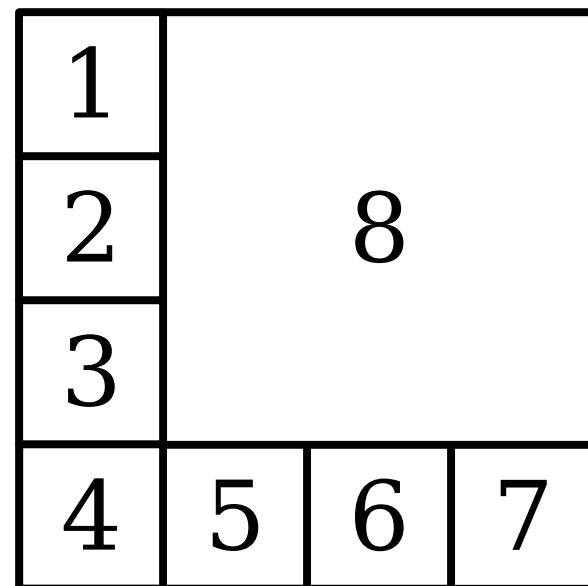
1 **2** **3** 4 **5** **6** 7 8 9 10 11 12



1 **2** **3** 4 **5** 6 **7** 8 9 10 11 12

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1 **2** **3** 4 **5** 6 7 8 9 10 11 12



1 **2** **3** 4 **5** 6 7 8 9 10 11 12

1	2	3
8	9	4
7	6	5

1 **2** **3** 4 **5** 6 7 8 9 **10** 11 12

1	2	
8	9	3
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	6	5

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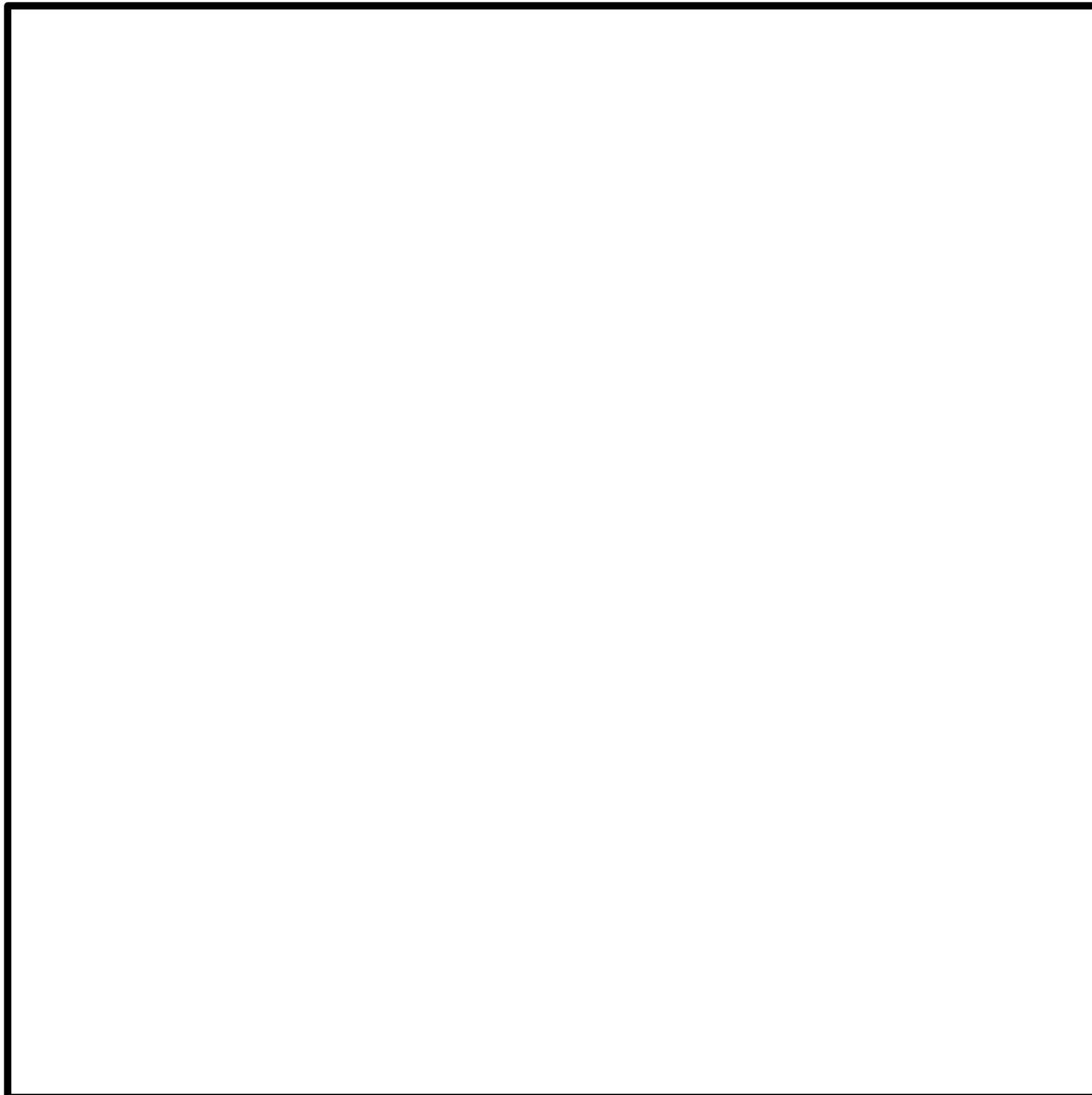
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3		11	8
4	5	6	7

1 **2** **3** 4 **5** 6 7 8 9 10 11 **12**

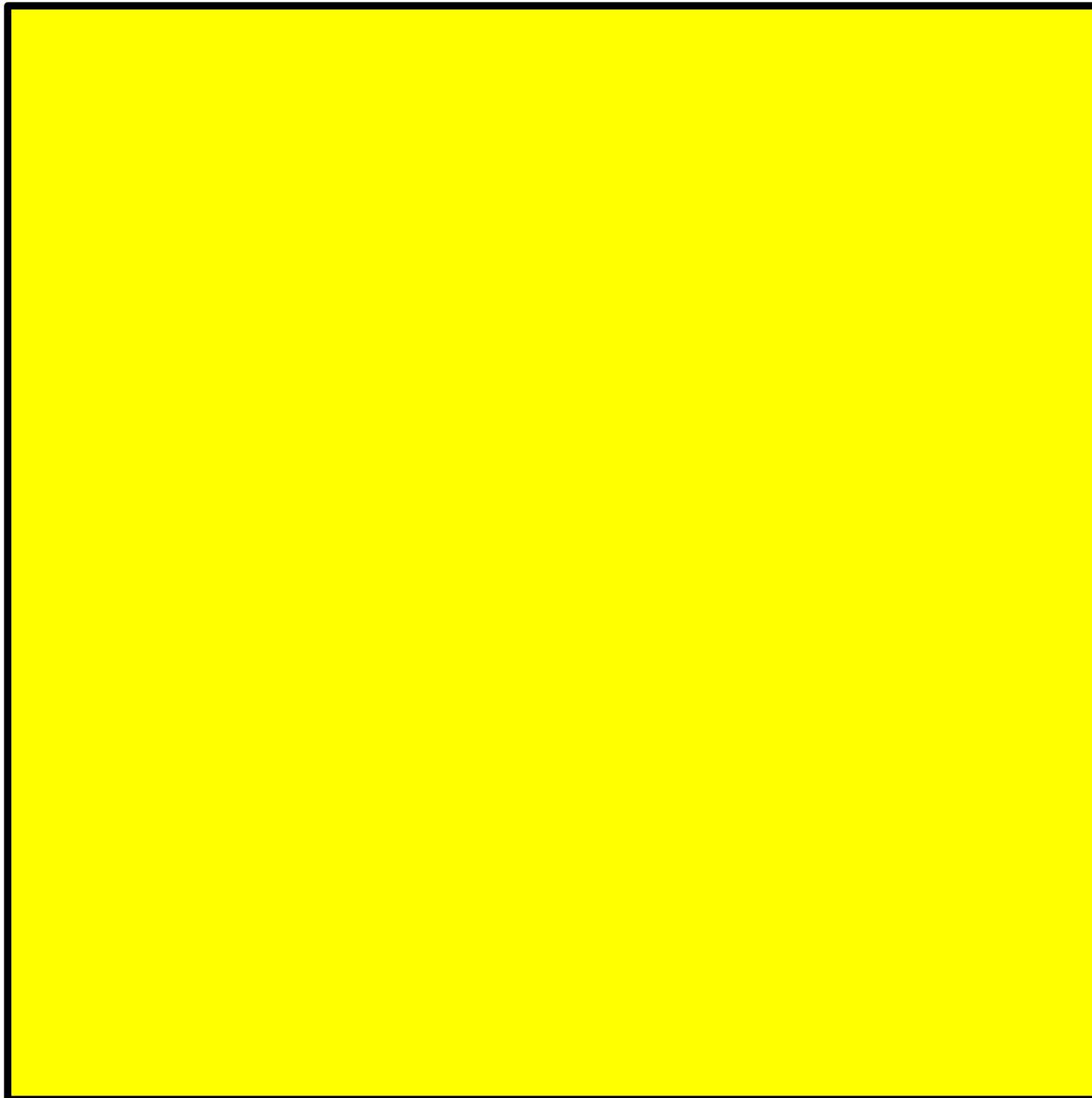
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12	11	4

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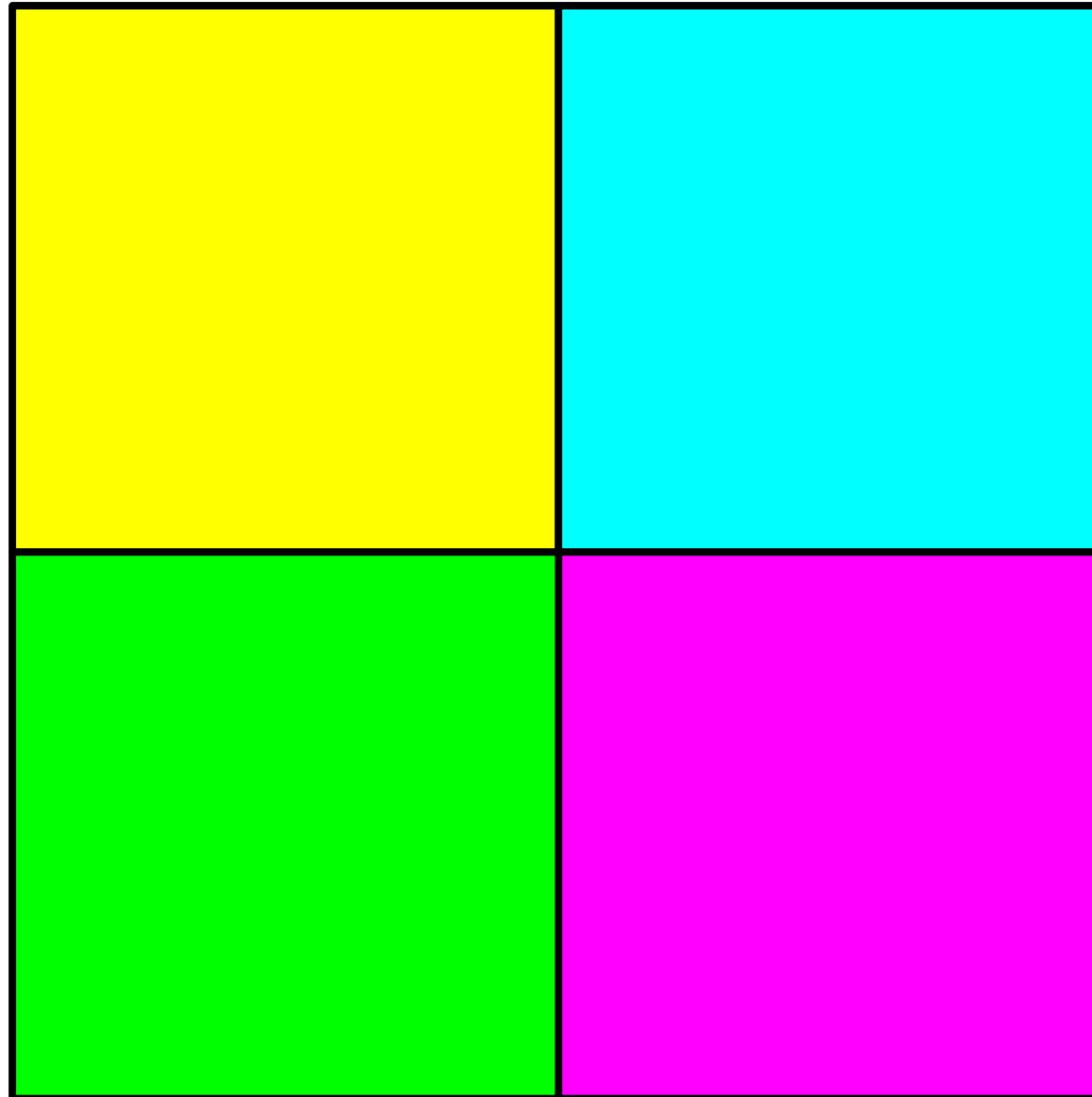
# An Insight



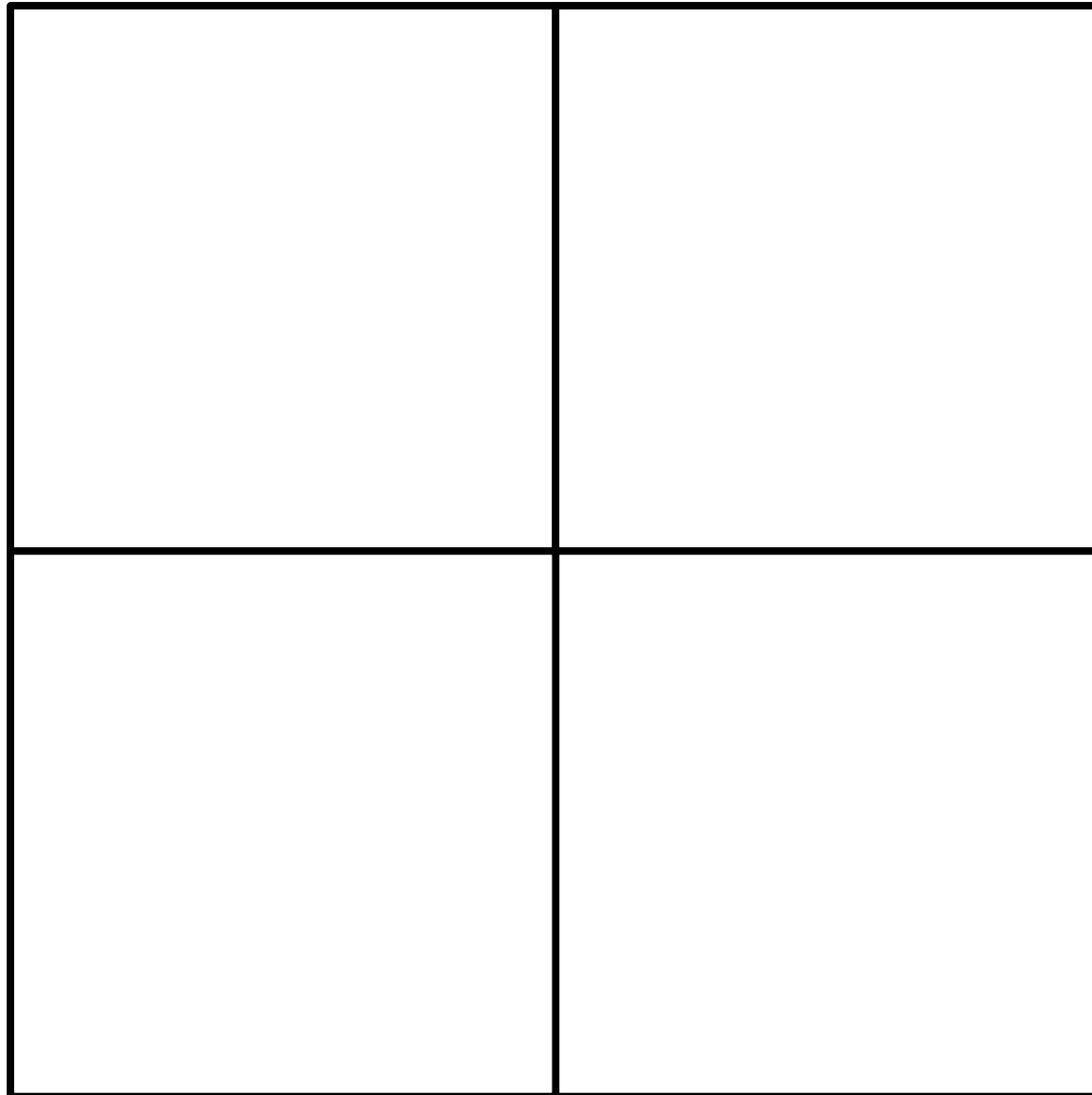
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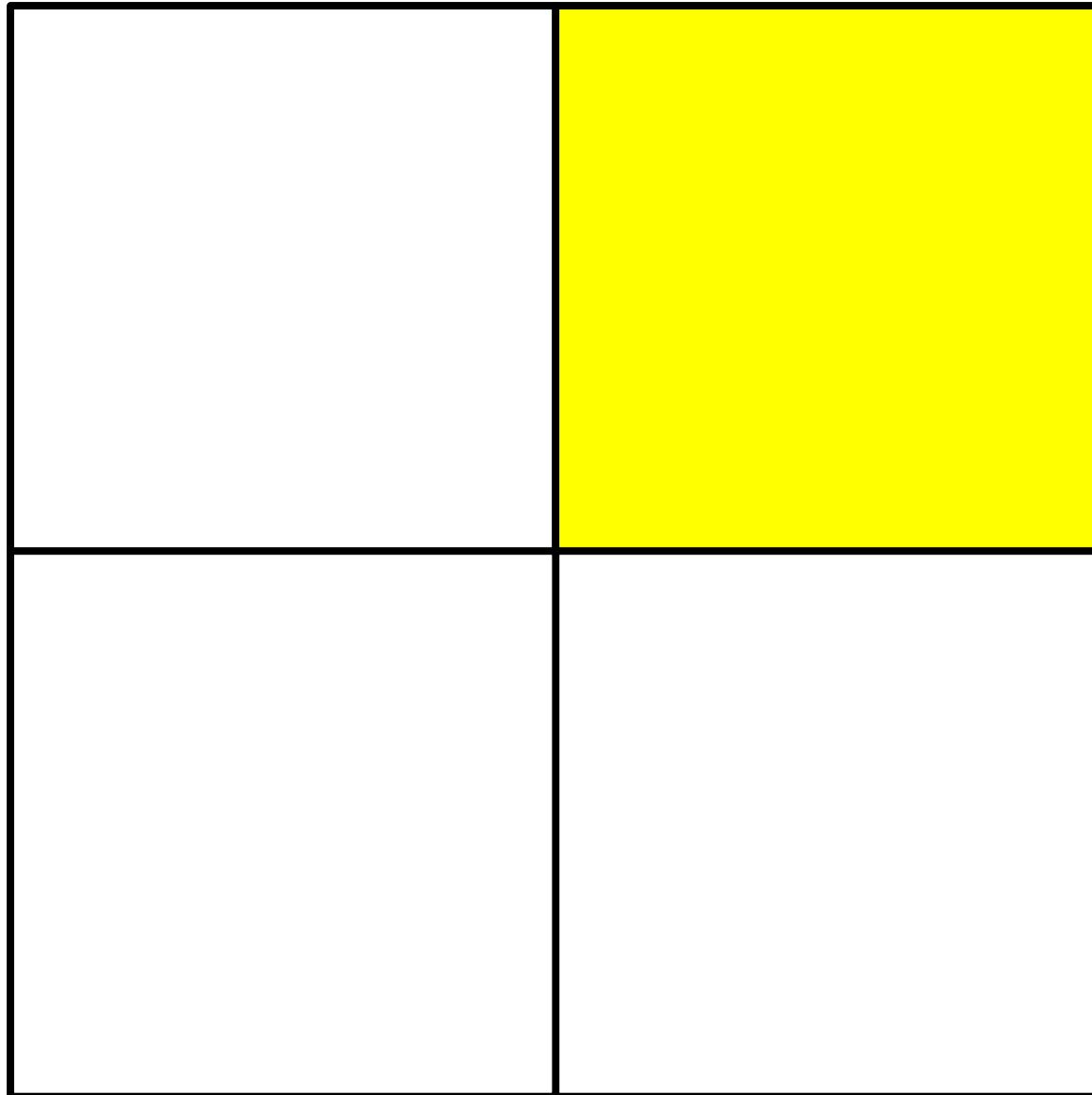
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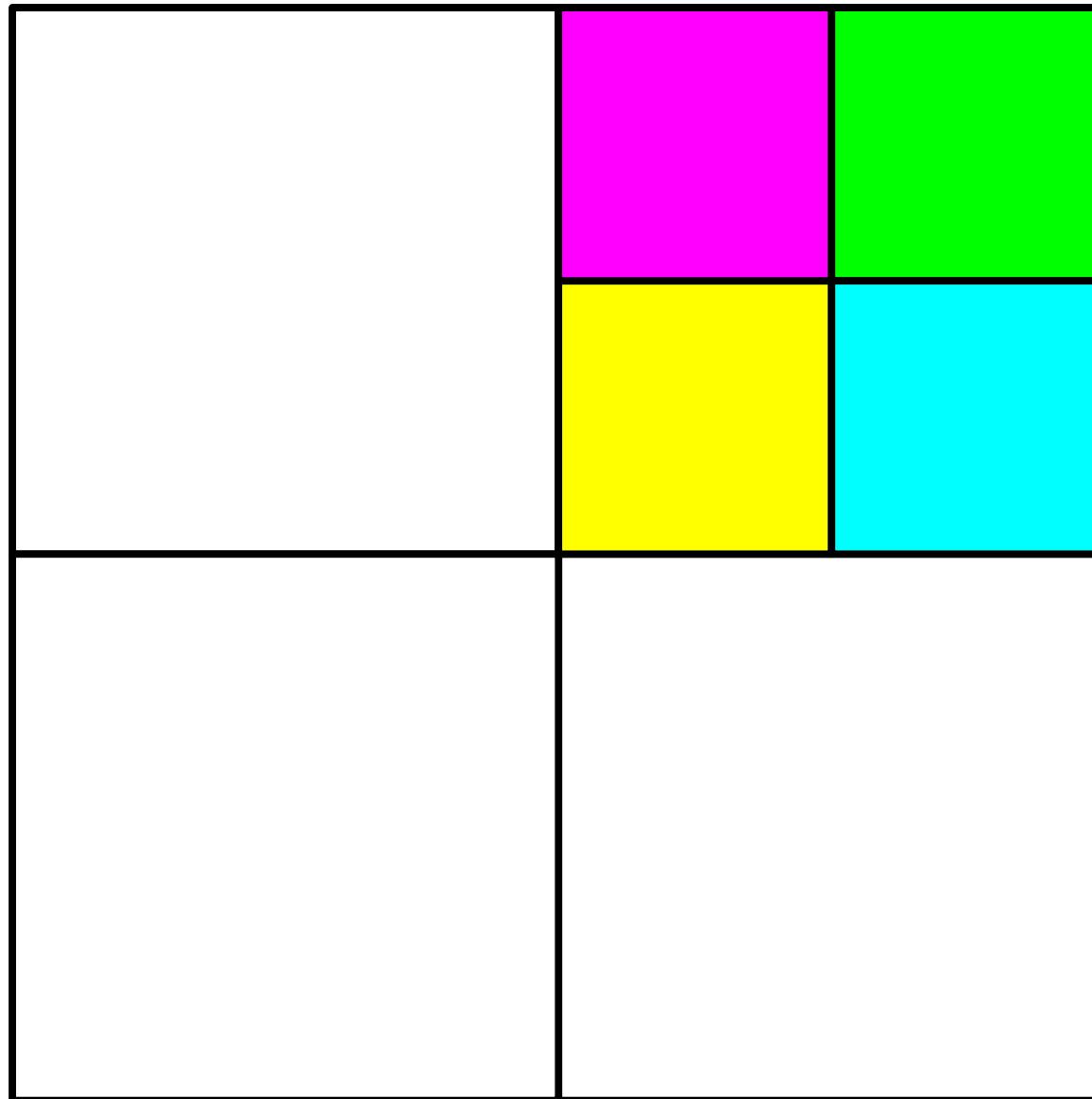
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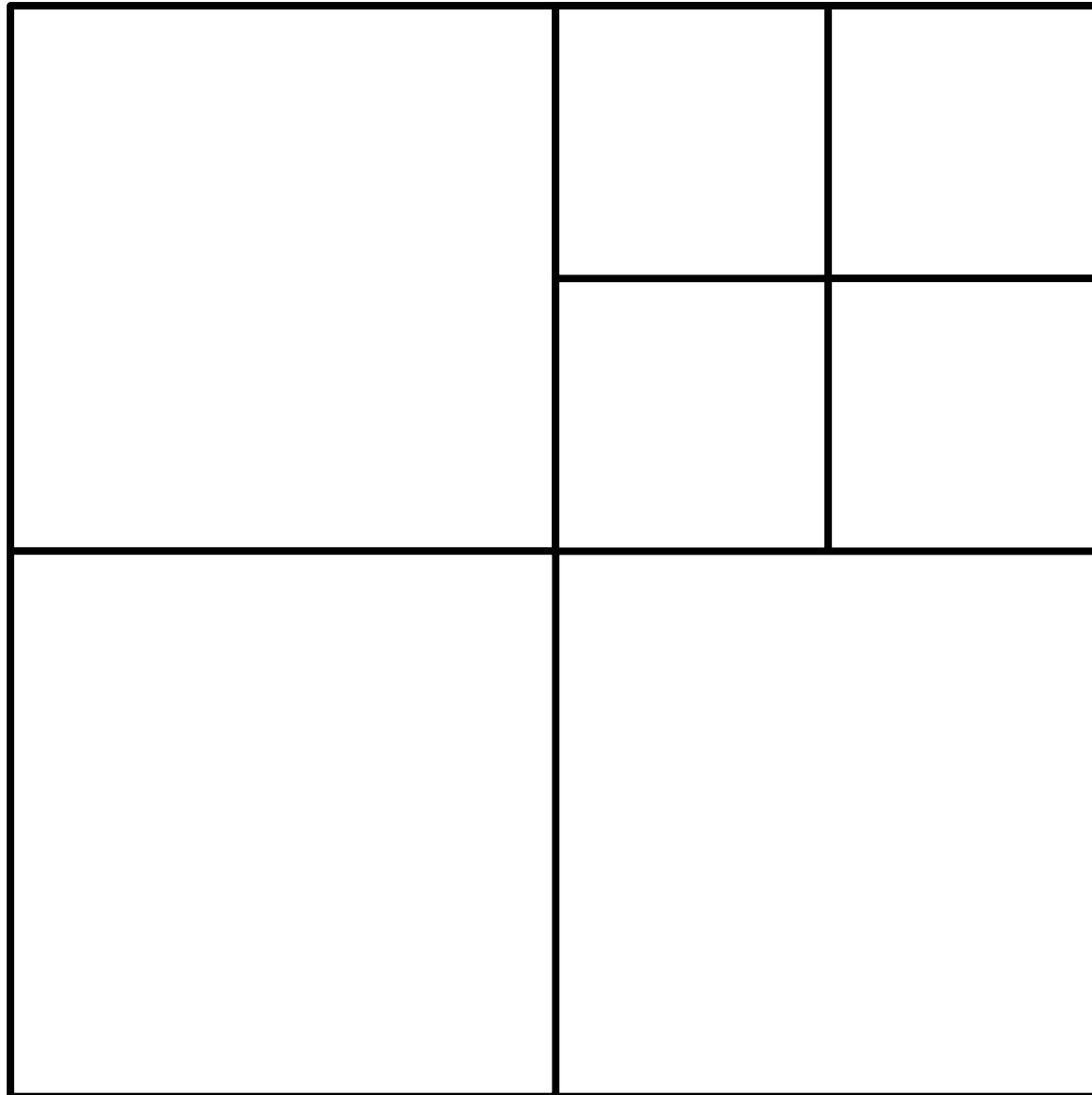
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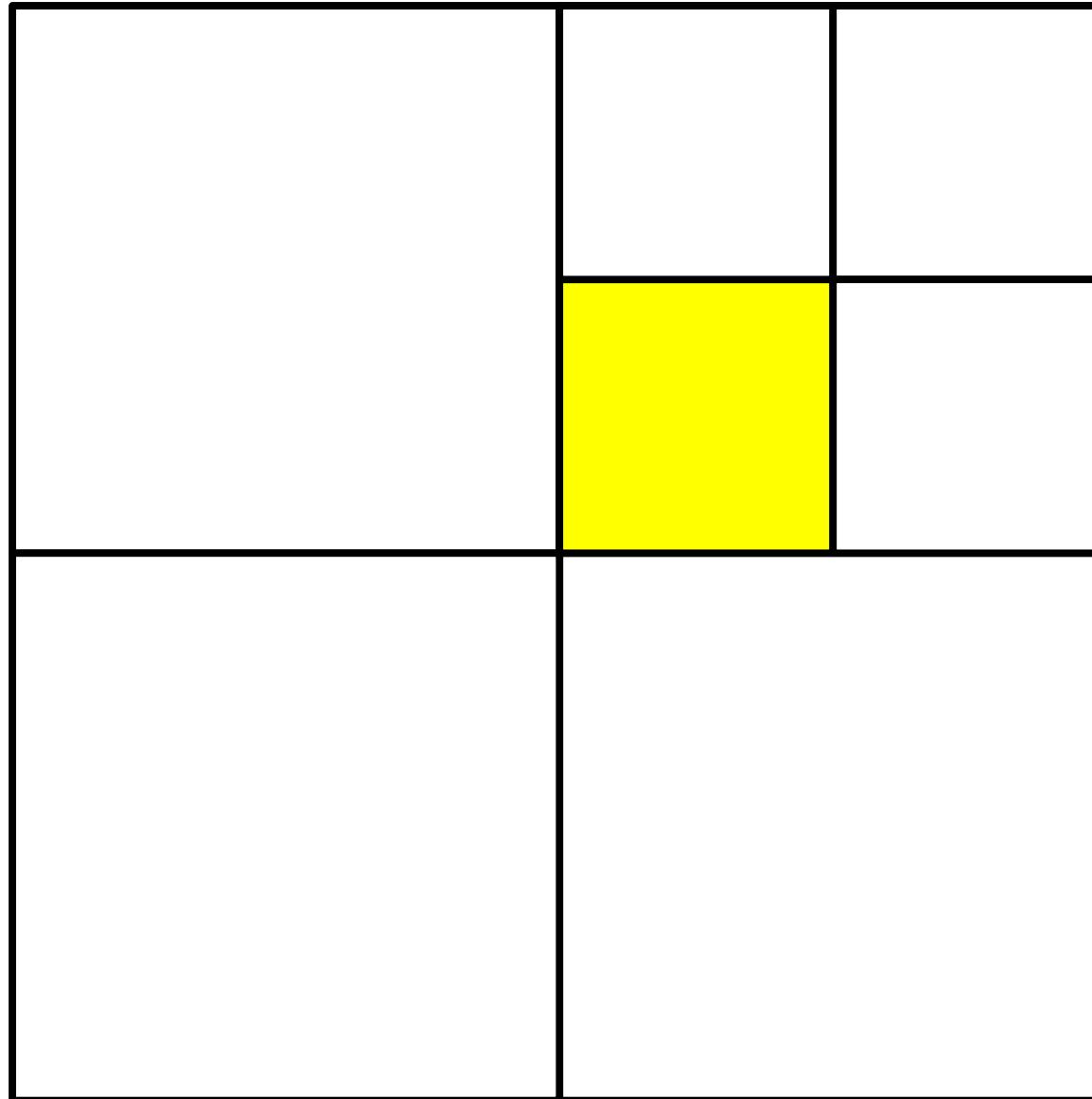
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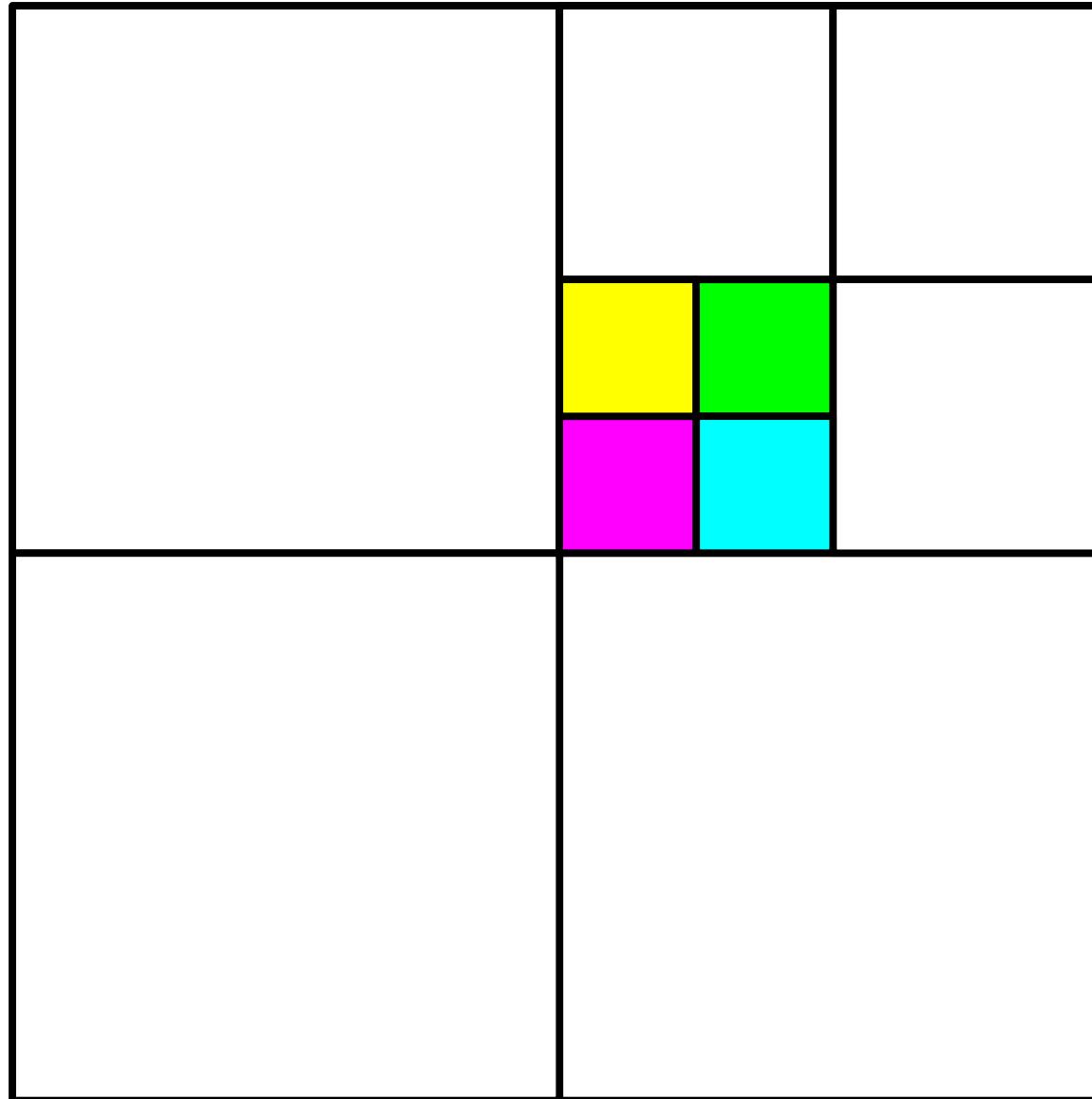
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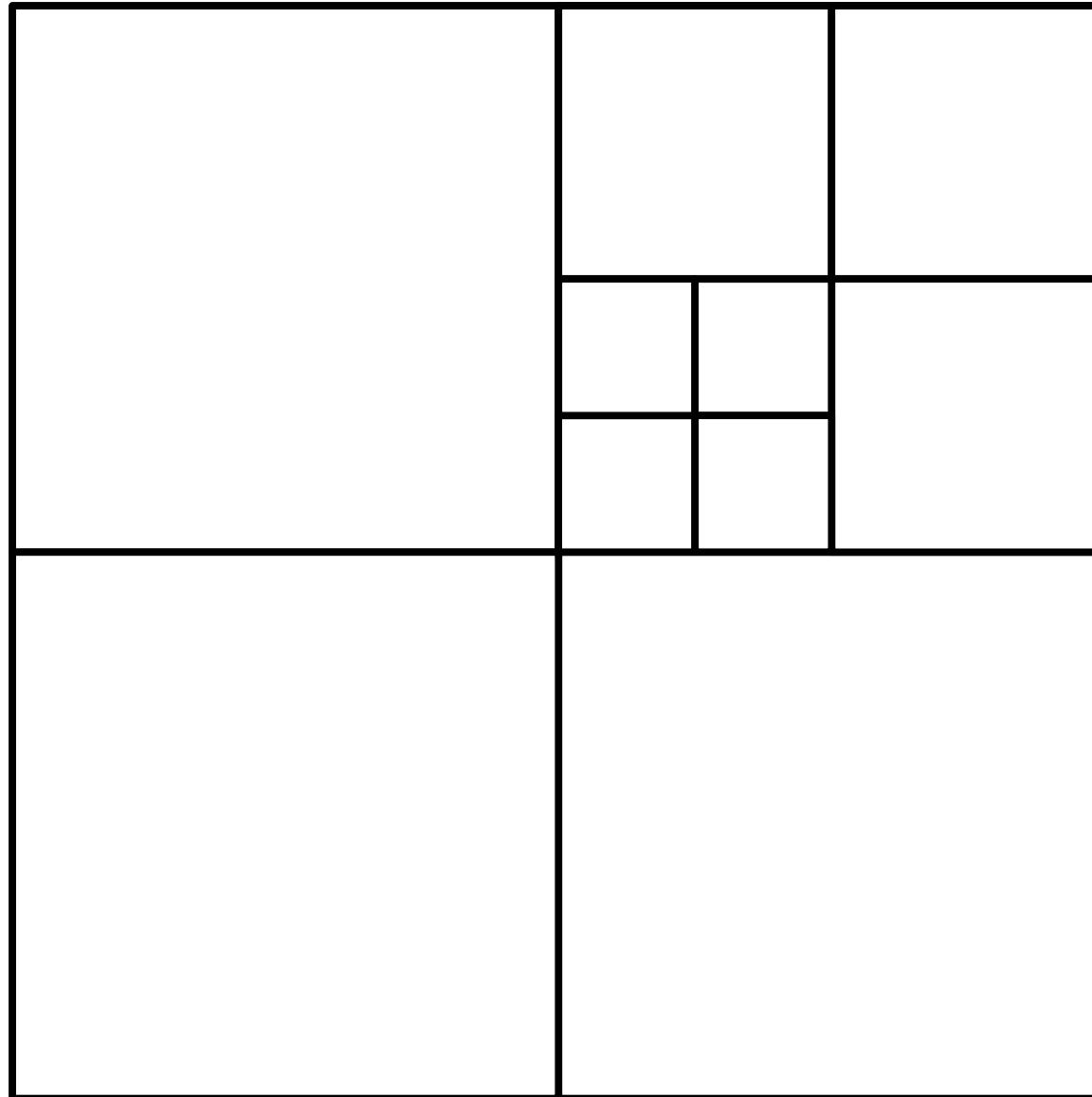
# An Insight



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3		
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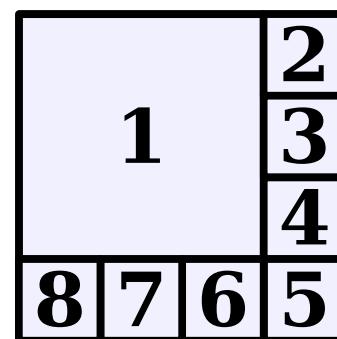
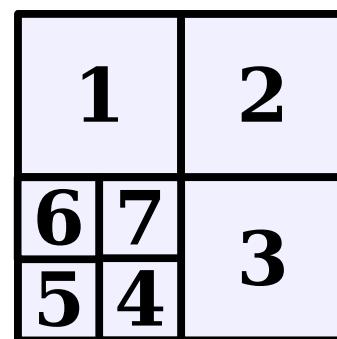
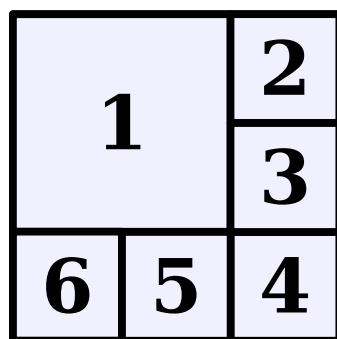
1	2	
6	7	3
5	4	

2			
3			
4			
1			
8	7	6	5

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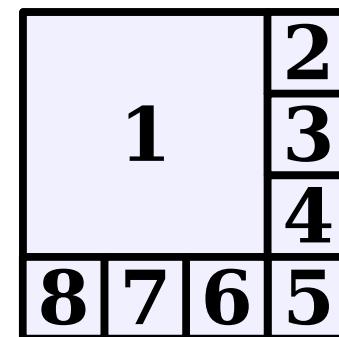
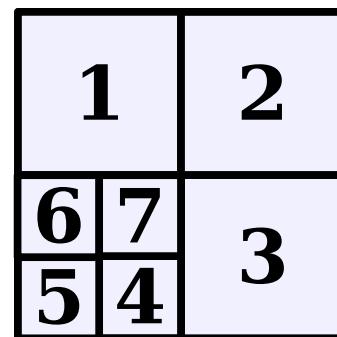
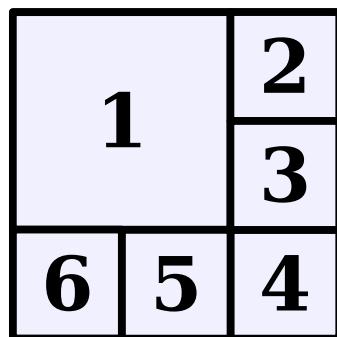


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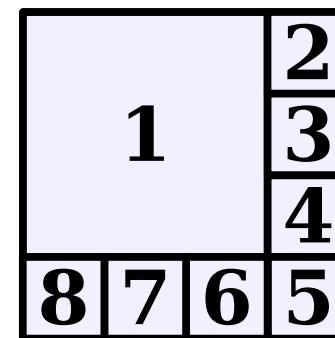
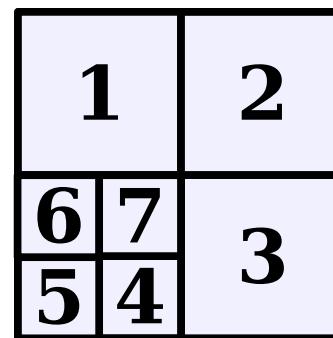
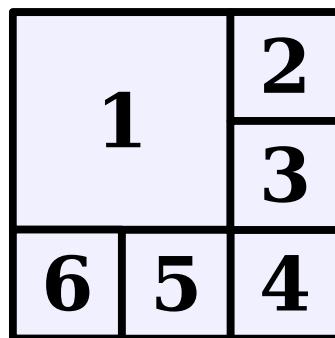


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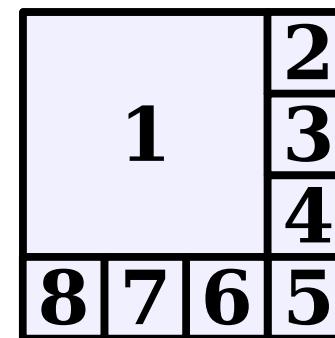
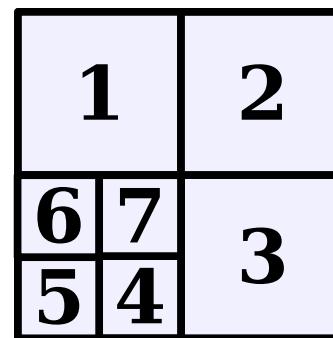
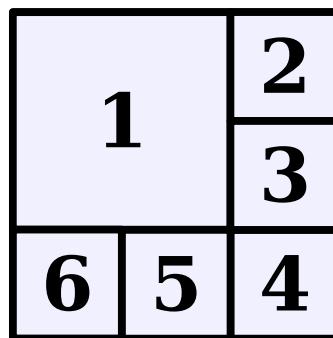


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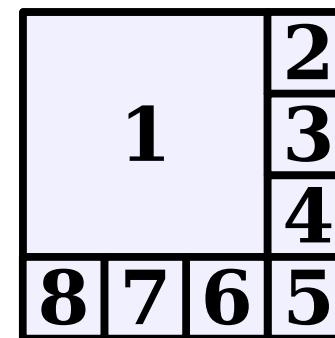
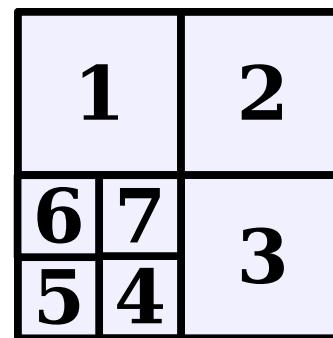
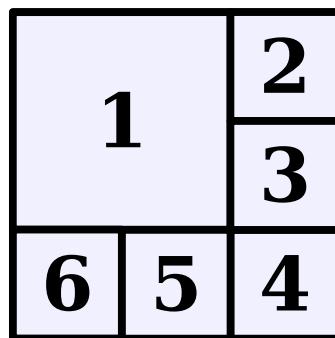


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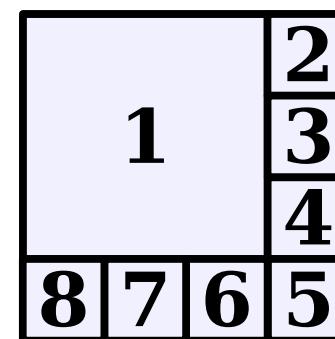
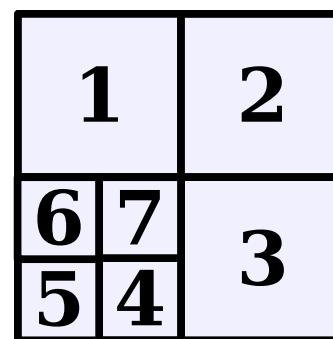
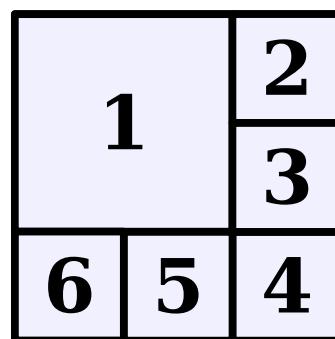


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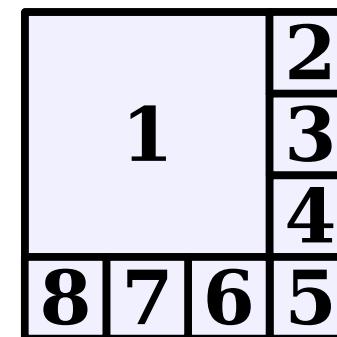
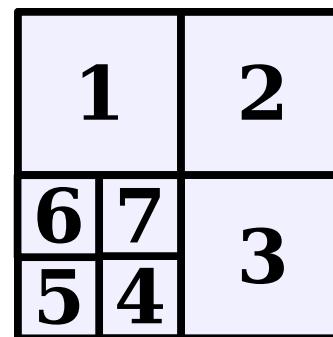
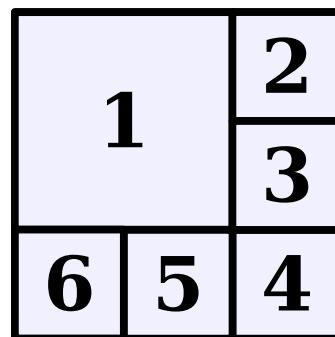


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# Generalizing Induction

- When doing a proof by induction,
  - feel free to use multiple base cases, and
  - feel free to take steps of sizes other than one.
- If you do, make sure that...
  - ... you actually need all your base cases. Avoid redundant base cases that are already covered by a mix of other base cases and your inductive step.
  - ... you cover all the numbers you need to cover. Trace out your reasoning and make sure all the numbers you need to cover really are covered.
- As with a proof by cases, you don't need to separately prove you've covered all the options. We trust you.

# More on Square Subdivisions

- There are a ton of interesting questions that come up when trying to subdivide a rectangle or square into smaller squares.
- In fact, one of the major players in early graph theory (William Tutte) got his start playing around with these problems.
- Good starting resource: this Numberphile video on [\*Squaring the Square\*](#).

# Next Time

- ***“Build Up” vs “Build Down”***
  - A subtle but key point in induction proofs.
- ***Complete Induction***
  - Expanding our inductive hypothesis.